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Ω-nets, Scott open sets and topologies on function spaces

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Introduction

In this paper we show that the basic results concerning the Scott open and closed sets, compact and bounded sets, compact-open and Isbell topologies, splitting and jointly continuous topologies and continuous convergence remain true if we replace these notions by the corresponding "Ω-notions", where Ω is a fixed class of directed sets. For each of the above notions it is possible to define different generalizations and the problem is to find the appropriate ones which satisfy the corresponding basic relations. We define "Ω-notions" using the nets (with directed sets from Ω) in the set of all closed subsets of a topological space.

In what follows, we denote by Ω a class of directed sets. A class Ω is called *cofinal closed* if every cofinal subset of an element of Ω belongs to Ω. A net in a space X is a map $S : \Lambda \to X$, where Λ is a directed set. The net S is also denoted by $\{x_\lambda, \lambda \in \Lambda\}$, where $x_\lambda = S(\lambda)$. If $\Lambda \in \Omega$, then this net is called Ω-net. If $\{A_\lambda, \lambda \in \Lambda\}$ is a net in the set $\mathcal{P}(Y)$ of all subsets of a space Y, then the *upper limit* of this net (see, for example, [K]), which is denoted by $\overline{\lim}_\Lambda(A_\lambda)$, is the set of all *cluster* points of Y, that is, the points y of Y such that for every $\lambda_0 \in \Lambda$ and for every open neighbourhood U of y in Y there exists an element $\lambda \in \Lambda,\ \lambda \geq \lambda_0$, for which $A_\lambda \cap U \neq \emptyset$.

We denote by $\mathcal{O}(Y)$ and $\mathcal{K}(Y)$ the set of all open and the set of all closed subsets of a space Y, respectively. For two spaces Y and Z we denote by $C(Y,Z)$ the set of all continuous maps of Y into Z. By $\mathbf{S}$ we denote the Sierpinski space, that is, the set $\{0,1\}$ with the topology $\tau(\mathbf{S}) \equiv \{\emptyset, \{0,1\}, \{0\}\}$. The sets $\mathcal{Q}(Y)$, $\mathcal{K}(Y)$ and $C(Y,\mathbf{S})$ can be identified as follows: every element U of $\mathcal{O}(Y)$ we identify with the element $Y \setminus U$

of $\mathcal{K}(Y)$ and with the element f of $C(Y, \mathbf{S})$ for which $f(U) \subseteq \{0\}$ and $f(Y \setminus U) \subseteq \{1\}$. Then for every topology on one of the above sets we can consider the corresponding topology on the other sets. By $\mathrm{Cl}(M)$ and $|M|$ we denote the closure and cardinality of a subset M of a space, respectively.

We identify an ordinal α with the set of all ordinals less than α and a cardinal β with the least ordinal of cardinality β.

I. Ω-Scott open sets and Ω-Bounded subsets

In this section we denote by Y a fixed topological space. The following definition of the Scott topology τ_s on the set $\mathcal{O}(Y)$ was given in [D-K]: A subset $\mathbb{H}$ of $\mathcal{O}(Y)$ is an element of τ_s if and only if: (α) $\mathbb{H}$ contains every open set of Y containing an element of $\mathbb{H}$, and (β) for every collection of open sets whose union belongs to $\mathbb{H}$ there are finitely many elements of this collection whose union also belongs to $\mathbb{H}$.

In [I-P] the characterization of Scott open and Scott closed subsets of $\mathcal{O}(Y)$ and $\mathcal{K}(Y)$ was given using the notion of the upper limit of a net in $\mathcal{K}(Y)$. Using these chraracterizations we can give the following definition.

1. Ω-Scott subsets. A subset $\mathbb{L}$ of $\mathcal{K}(Y)$ is called Ω-*Scott open* if the following conditions are true: (α) if $L \in \mathbb{L}$, $K \in \mathcal{K}(Y)$ and $K \subseteq L$, then $K \in \mathbb{L}$, and (β) if $\{K_\lambda,\ \lambda \in \Lambda\}$ is an Ω-net in $\mathcal{K}(Y)$ and $\overline{\lim}_\Lambda(K_\lambda) \in \mathbb{L}$, then there exists an element $\lambda_0 \in \Lambda$ such that $K_\lambda \in \mathbb{L}$ for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$.

It is easy to verify that the set of all Ω-Scott open subsets is a topology on the set $\mathcal{K}(Y)$. This topology is called Ω-*Scott topology*.

1.1. Theorem. *The following statements are true:*

(1) A subset $\mathbb{F}$ of $\mathcal{K}(Y)$ is Ω-Scott closed if and only if: (α) if $K \in \mathbb{F}$ and $K \subseteq L \in \mathcal{K}(Y)$, then $L \in \mathbb{F}$, and (β) if $\{K_\lambda,\ \lambda \in \Lambda\}$ is an Ω-net $\mathcal{K}(Y)$ and $K_\lambda \in \mathbb{F}$ for every $\lambda \in \Lambda$, then $\overline{\lim}_\Lambda(K_\lambda) \in \mathbb{F}$.

(2) A subset $\mathbb{H}$ of $\mathcal{O}(Y)$ is Ω-Scott open if and only if: (α) if $V \in \mathbb{H}$ and $V \subseteq U \in \mathcal{O}(Y)$, then $U \in \mathbb{H}$, and (β) if $\{V_\lambda,\ \lambda \in \Lambda\}$ is an Ω-net in $\mathcal{O}(Y)$ and $Y \setminus \overline{\lim}_\Lambda(Y \setminus V_\lambda) \in \mathbb{H}$, then there exists an element $\lambda_0 \in \Lambda$ such that $V_\lambda \in \mathbb{H}$ for every $\lambda \in \Lambda,\ \lambda \geq \lambda_0$.

(3) A subset $\mathbb{K}$ of $\mathcal{O}(Y)$ is Ω-Scott closed if and only if: (α) if $U \in \mathbb{K}$, $V \in \mathcal{O}(Y)$ and $V \subseteq U$, then $V \in \mathbb{K}$, and (β) if $\{V_\lambda,\ \lambda \in \Lambda\}$ is an Ω-net in $\mathcal{O}(Y)$ and $V_\lambda \in \mathbb{K}$ for every $\lambda \in \Lambda$, then $Y \setminus \overline{\lim}_\Lambda(Y \setminus V_\lambda) \in \mathbb{K}$.

The proof of this theorem is similar to the proof of Theorem 10 of [I–P].

2. (α, β)**-Scott open sets.** In this section we give another natural generalization of the notion of a Scott open set and prove that this generalization, "almost always", is a particular case of the notion of an Ω-Scott open set.

Let α be an infinite cardinal and β be a cardinal or symbol ∞ such that $\alpha < \beta$. (We suppose that $\gamma \neq \infty$ and $\gamma < \infty$ for every cardinal γ). A subset $\mathbb{H}$ of $\mathcal{O}(Y)$ is called (α, β)-*Scott open* if: (α) $\mathbb{H}$ contains every open set of Y containing an element of $\mathbb{H}$, and (β) for every collection $\{U_\mu : \mu \in M\}$ of open sets of Y, whose union belongs to $\mathbb{H}$, where $|M| < \beta$, there exists a subset $N \subseteq M$ with $|N| < \alpha$ such that $\bigcup\{U_\mu : \mu \in N\} \in \mathbb{H}$.

As for the Scott open sets, the set of all (α, β)-Scott open sets of $\mathcal{O}(Y)$ is a topology calling (α, β)-*Scott topology* on $\mathcal{O}(Y)$. It is easy to see that the Scott topology coincides with the (ω, ∞)-Scott topology on $\mathcal{O}(Y)$.

We observe that a subset $\mathbb{L}$ of $\mathcal{K}(Y)$ is (α, β)-Scott open if and only if: (α) if $L \in \mathbb{L}$, $K \in \mathcal{K}(Y)$ and $K \subseteq L$, then $K \in \mathbb{L}$, and (β) if for a collection $\{K_\mu : \mu \in M\}$ of elements of $\mathcal{K}(Y)$, where $|M| < \beta$, we have $\bigcap\{K_\mu : \mu \in M\} \in \mathbb{L}$, then there exists a subset $N \subseteq M$ with $|N| < \alpha$ such that $\bigcap\{K_\mu : \mu \in N\} \in \mathbb{L}$.

2.1. Theorem. *Suppose that α is regular and for every $\gamma < \beta$ cardinality of the set of all subsets of the set γ of cardinality less than α, is less than or equal to β. Then there exists a class Ω of directed sets such that the Ω-Scott topology coincides with (α, β)-Scott topology on $\mathcal{K}(Y)$.*

Proof. Let Ω be the class of all directed sets Λ with $|\Lambda| < \beta$ having the property: for every subset $\Lambda' \subseteq \Lambda$ with $|\Lambda'| < \alpha$ there exists an element $\lambda_0 \in \Lambda$ such that $\lambda \leq \lambda_0$ for every $\lambda \in \Lambda'$. We prove that (α, β)-Scott topology coincides with the Ω-Scott topology on $\mathcal{K}(Y)$.

Let $\mathbb{L}$ be an (α, β)-Scott open set. We prove that $\mathbb{L}$ is Ω-Scott open. Obviously, it is sufficient to prove property (β) of Section 1.

Let $\{K_\lambda, \lambda \in \Lambda\}$ be an Ω-net in $\mathcal{K}(Y)$ and let $\overline{\lim}_\Lambda(K_\lambda) \in \mathbb{L}$. We must prove that there exists an element $\lambda_0 \in \Lambda$ such that $K_\lambda \in \mathbb{L}$ for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. For every $\lambda \in \Lambda$ we set $K'_\lambda = \mathrm{Cl}(\bigcup\{K'_{\lambda'} : \lambda' \geq \lambda\})$. Obviously $K'_{\lambda_1} \subseteq K'_{\lambda_2}$ if $\lambda_2 \leq \lambda_1$. It is also easy to verify that $\overline{\lim}_\Lambda(K'_\lambda) = \bigcap\{K'_\lambda : \lambda \in \Lambda\}$ and $\overline{\lim}_\Lambda(K'_\lambda) = \overline{\lim}_\Lambda(K_\lambda)$. Hence $\overline{\lim}_\Lambda(K'_\lambda = \bigcap\{K'_\lambda : \lambda \in \Lambda\} \in \mathbb{L}$. Since $\mathbb{L}$ is (α, β)-Scott open and $|\Lambda| < \beta$ there exists a subset $\Lambda' \subseteq \Lambda$ with $|\Lambda'| < \alpha$ such that $\bigcap\{K'_\lambda : \lambda \in \Lambda'\} \in \mathbb{L}$. Since $\lambda \in \Omega$, there exists an element $\lambda_0 \in \Lambda$ such that $\lambda \leq \lambda_0$ for every $\lambda \in \Lambda'$. This means that $K'_{\lambda_0} \subseteq \bigcap\{K_\lambda : \lambda \in \Lambda'\}$. Hence $K'_{\lambda_0} \in \mathbb{L}$. Since $K_\lambda \subseteq K'_{\lambda_0}$ for every $\lambda \geq \lambda_0$ we have $K_\lambda \in \mathbb{L}$. Thus $\mathbb{L}$ is Ω-Scott open.

Conversely, let $\mathbb{L}$ be an Ω-Scott open subset of $\mathcal{K}(Y)$. We prove that $\mathbb{L}$ is (α, β)-Scott open. It is sufficient to prove that if $\{K_\mu : \mu \in M\}$ is a collection of elements of $\mathcal{K}(Y)$, where $|M| < \beta$ and $\bigcap\{K_\mu : \mu \in M\} \in \mathbb{L}$, then there exists a subset $N \subseteq M$ with $|N| < \alpha$ such that $\bigcap\{K_\mu : \mu \in N\} \in \mathbb{L}$. Let Λ be the set of all subsets of M of cardinality less than α directed by inclusion. By conditions of the theorem, $\Lambda \in \Omega$. Consider the net $\{K_\lambda, \lambda \in \Lambda\}$ in $\mathcal{K}(Y)$, where $K_\lambda = \bigcap\{K_\mu : \mu \in \lambda\}$. It is easy to verify that $\overline{\lim_\Lambda}(K_\lambda) = \bigcap\{K_\lambda : \lambda \in \Lambda\} = \bigcap\{K_\mu : \mu \in M\}$. Hence $\overline{\lim_\Lambda}(K_\lambda) \in \mathbb{L}$. Since $\mathbb{L}$ is Ω-Scott open and $\Lambda \in \Omega$, there exists $\lambda_0 \in \Lambda$ such that $K_\lambda \in \mathbb{L}$ for every $\lambda \in \lambda,\ \lambda \geq \lambda_0$. Setting $N = \lambda_0$ we have $|N| < \alpha$ and $K_\lambda = \bigcap\{K_\mu : \mu \in N\} \in \mathbb{L}$. Thus $\mathbb{L}$ is (α, β)-Scott open.

2.2. Problem. Is Theorem 2.1 true for every infinite (regular) cardinal α and cardinal β for which $\alpha < \beta$?

We observe that under hypothesis **CH** this theorem is true if we suppose only that α is regular.

3. Ω-Bounded subsets. A subset B of Y is called *bounded* if and only if for every open cover of Y there exists a finite subcollection of this cover, the union of elements of which contains the set B.

Let α be an infinite cardinal and β be a cardinal or symbol ∞ such that $\alpha < \beta$. A subset B of Y is called (α, β)-*bounded* if for every open cover $\mathcal{U}$ of Y which $|\mathcal{U}| < \beta$ there exists a subfamily of $\mathcal{U}$ of cardinality less than α covering the set B. About the notions boundedness and (α, β)-boundedness see, for example [$\mathbf{L}_1$]. (We observe that, in general, the above notion of (α, β)-bounded set does not coincide (for limit α and β) with the notion of (α, β)-bounded set given in [$\mathbf{L}_1$]).

3.1. Theorem. *A subset B of Y is bounded if and only if for every net $\{K_\lambda, \lambda \in \Lambda\}$ in $\mathcal{K}(Y)$ such that $\overline{\lim_\Lambda}(K_\lambda) = \emptyset$, there exists an element $\lambda_0 \in \Lambda$ for which $K_\lambda \cap B = \emptyset$ for every $\lambda \in \Lambda, \lambda \geq \lambda_0$.*

Proof. Let B be a bounded subset of Y and let $\{K_\lambda, \lambda \in \Lambda\}$ be a net in $\mathcal{K}(Y)$ such that $\overline{\lim_\Lambda}(K_\lambda) = \emptyset$. Then for every $y \in Y$ there exist an open neighbourhood U_y of y in Y and an element $\lambda_y \in \Lambda$ such that $K_\lambda \cap U_y = \emptyset$ for every $\lambda \geq \lambda_y$.

Since B is bounded and $Y = \bigcup\{U_y : y \in Y\}$, there exist $y_1, \ldots, y_n \in Y$ such that

$$B \subseteq \bigcup\{U_{y_i} : i = 1, \ldots, n\}.$$

Let λ_0 be an element of Λ such that $\lambda_0 \geq \lambda_{y_i}$ for every $i = 1, \ldots, n$. Then for every $\lambda \geq \lambda_0$, we have $U_{y_i} \subseteq Y \setminus K_\lambda$ and hence

$$B \subseteq \bigcup\{U_{y_i} : i = 1, \ldots, n\} \subseteq Y \setminus K_\lambda.$$

Thus $K_\lambda \cap B = \emptyset$ for every $\lambda \in \Lambda,\ \lambda \geq \lambda_0$.

Conversely, suppose that the subset B satisfies the condition of the theorem. We prove that B is bounded. Let $\mathcal{U}$ be an open cover of the space Y. Let Λ be the set of all finite subsets of $\mathcal{U}$ directed by inclusion and let $\{K_\lambda, \lambda \in \Lambda\}$ be the net in $\mathcal{K}(Y)$ for which $Y \setminus K_\lambda$ is the union of elements of λ. Obviously $K_{\lambda_1} \subseteq K_{\lambda_2}$ if $\lambda_2 \subseteq \lambda_1$. From this it follows that $\overline{\lim}_\Lambda(K_\lambda) = \bigcap\{K_\lambda : \lambda \in \Lambda\}$.

Since $\bigcap\{K_\lambda : \lambda \in \Lambda\} = Y \setminus (\bigcup\{Y \setminus K_\lambda : \lambda \in \Lambda\}) = Y \setminus (\bigcup\{U : U \in \mathcal{U}\}) = \emptyset$, we have $\overline{\lim}_\Lambda(K_\lambda) = \emptyset$. By assumption there exists an element $\lambda_0 \in \Lambda$ for which $K_\lambda \cap B = \emptyset$ for every $\lambda \in \lambda,\ \lambda \geq \lambda_0$. Hence $B \subseteq Y \setminus K_{\lambda_0} = \bigcup\{U : U \in \lambda_0\}$. Thus B is bounded.

In particular, from Theorem 3.1 we have: **A** space X is compact if and only if for every net $\{K_\lambda, \lambda \in \Lambda\}$ in $\mathcal{K}(X)$ such that $\overline{\lim}_\Lambda(K_\lambda) = \emptyset$ there exists an element $\lambda_0 \in \Lambda$ for which $K_\lambda = \emptyset$ for every $\lambda \in \Lambda,\ \lambda \geq \lambda_0$.

Using Theorem 3.1 we can give the following definitions.

3.2. Definitions. A subset B of Y is called Ω-*bounded in* Y if for every Ω-net $\{K_\lambda, \lambda \in \Lambda\}$ in $\mathcal{K}(Y)$ such that $\overline{\lim}_\Lambda(K_\lambda) = \emptyset$ there exists an element $\lambda_0 \in \Lambda$ for which $K_\lambda \cap B = \emptyset$ for every $\lambda \in \Lambda, \lambda \geq \lambda_0$.

A space X is called Ω-*compact* if for every Ω-net $\{K_\lambda, \lambda \in \Lambda\}$ in $\mathcal{K}(X)$ such that $\overline{\lim}_\Lambda(K_\lambda) = \emptyset$ there exists an element $\lambda_0 \in \Lambda$ for which $K_\lambda = \emptyset$ for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. (We use the same term as in [V] while, in general, the above notion of Ω-compact space does not coincide with the notion of Ω-compact space given in [V]. However, it is not difficult to prove that these notions coincide if Ω is cofinal closed).

By Theorem 3.1 it follows that if Ω is the class of all directed sets, then the notions of Ω-boundedness and Ω-compactness coincide with the notions boundedness and compactness, respectively. In general, the Ω-boundedness preserves many properties of the boundedness. Such properties, for example, are the following:

(1) A subset B of Y is Ω-bounded if and only if B is Ω-bounded in $\mathrm{Cl}(B)$.
(2) If B is closed and Ω-bounded in Y, then B is Ω-compact.
(3) If B is Ω-bounded in Y and $B' \subseteq B$, then the subset B' is Ω-bounded.
(4) If $B_1, \dots, B_n$ is Ω-bounded in Y, then the subset $\bigcup\{B_i : i = 1, \dots, n\}$ is also Ω-bounded.
(5) If the subset B of Y is Ω-compact then B is Ω-bounded.
(6) By properties (3) and (4) it follows that the notion of Ω-boundedness is a boundedness in the since of [H].

3.3. Theorem. *Let K be an Ω-compact subset of Y. Then the set*

$$\mathbb{H} \equiv \{U \in \mathcal{O}(Y) : K \subseteq U\},$$

is an Ω-Scott open subset of $\mathcal{O}(Y)$.

Proof. Obviously, if $V \in \mathbb{H}$ and $V \subseteq U \in \mathcal{O}(Y)$, then $U \in \mathbb{H}$. Let $\{V_\lambda, \lambda \in \Lambda\}$ be an Ω-net in $\mathcal{O}(Y)$ and $Y \setminus \overline{\lim}_\Lambda(Y \setminus V_\lambda) \in \mathbb{H}$. Then,

$K \subseteq Y \setminus \overline{\lim}_\Lambda(Y \setminus V_\lambda)$, that is, $K \cap \overline{\lim}_\Lambda(Y \setminus V_\lambda) = \emptyset$. Let $K_\lambda = K \cap (Y \setminus V_\lambda), \lambda \in \Lambda$. Then $\{K_\lambda, \lambda \in \Lambda\}$ is a net in $\mathcal{K}(K)$. Obviously, $\overline{\lim}_\Lambda(K_\lambda) \subseteq K \cap \overline{\lim}_\Lambda(Y \setminus V_\lambda)$ (the first upper limit is considered in the space K and the second in Y) and hence $\overline{\lim}_\Lambda(K_\lambda) = \emptyset$. Since the space K is Ω-compact there exists an element $\lambda_0 \in \Lambda$ such that $K_\lambda = \emptyset$ for every $\lambda \geq \lambda_0$. Hence $K \subseteq V_\lambda$ and $V_\lambda \in \mathbb{H}$ for every $\lambda \geq \lambda_0$. By Theorem 1.1, $\mathbb{H}$ is an Ω-Scott open set of $\mathcal{O}(Y)$.

3.4. Theorem. *Let α and β be as in Theorem 2.1. Then there exists a class Ω of directed sets such that a subset B of a space X is (α, β)-bounded if and only if it is Ω-bounded.*

The proof of this theorem is similar to the proof of Theorem 3.1.

We observe that if α and β are as in Theorem 3.4, then there exists a class Ω of directed sets such that a space X is (α, β)-compact if and only if it is Ω-compact.

4. Strong Ω-Scott sets. We recall that a subset $\mathbb{H}$ of $\mathcal{O}(Y)$ is called *strong Scott open* (see [L-P]) if the following conditions are satisfied: (α) if $U \in \mathbb{H}$ and $U \subseteq V \in \mathcal{O}(Y)$, then $V \in \mathbb{H}$ and (β) for every open cover of Y, there exists finitely many elements of this cover, whose union belongs to $\mathbb{H}$. The set of all strong Scott open sets is a topology a calling *strong Scott topology* on the set $\mathcal{O}(Y)$. (See [L-P].)

We observe that a subset $\mathbb{L}$ of the set $\mathcal{K}(Y)$ belongs to the strong Scott topology on this set if and only if: (α) if $L \in \mathbb{L}$, $K \in \mathbb{L}$ and $K \subseteq L$, then $K \in \mathbb{L}$, and (β) if for a collection $\{K_\mu : \mu \in M\}$ of elements of $\mathcal{K}(Y)$ we have $\bigcap\{K_\mu : \mu \in M\} = \emptyset$, then there exists a finite subcollection, the intersection of elements of which belongs to $\mathbb{L}$.

As in Section 2 we give another natural generalization of the notion of strong Scott open set.

Let α and β be as in Section 2. A subset $\mathbb{H}$ of $\mathcal{O}(Y)$ is called *strong (α, β)-Scott open* if the following conditions are satisfied: (α) if $U \in \mathbb{H}$ and $U \subseteq V \in \mathcal{O}(Y)$, then $V \in \mathbb{H}$, and (β) for every open cover of Y of cardinality less than β, there exists a subcollection of cardinality less than α, the union of elements of which belongs to $\mathbb{H}$. Obviously, the set of all strong (α, β)-Scott open sets is a topology calling *Strong (α, β)-Strong topology* on $\mathcal{O}(Y)$.

4.1. Theorem. *A subset $\mathbb{L}$ of $\mathcal{K}(Y)$ is open in the strong Scott topology if and only if the following conditions are satisfied: (α) if $L \in \mathbb{L}$, $K \in \mathcal{K}(Y)$ and $K \subseteq L$, then $K \in \mathbb{L}$, and (β) if $\{K_\lambda, \lambda \in \Lambda\}$ is a net in $\mathcal{K}(Y)$ and $\overline{\lim}_\Lambda(K_\lambda) = \emptyset$, then there exists an element $\lambda_0 \in \Lambda$ such that $K_\lambda \in \mathbb{L}$ for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$.*

The proof of this theorem is similar to the proof of Theorem 3.1.

Using Theorem 4.1 we can give the following definition.

4.2. Definition. A subset $\mathbb{L}$ of $\mathcal{K}(Y)$ is called *strong Ω-Strong open* if the following conditions are satisfied: (α) if $L \in \mathbb{L}$, $K \in \mathcal{K}(Y)$ and $K \subseteq L$, then $K \in \mathbb{L}$, and (β) if $\{K_\lambda, \lambda \in \Lambda\}$ is an Ω-net in $\mathcal{K}(Y)$ and $\overline{\lim_\Lambda}(K_\lambda) = \emptyset$, then there exists an element $\lambda_0 \in \Lambda$ such that $K_\lambda \in \mathbb{L}$ for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$.

We observe that the set of all strong Ω-Scott open sets is a topology calling *strong Ω-Scott topology* on $\mathcal{K}(Y)$. Obviously, the strong Ω-Scott topology on $\mathcal{O}(Y)$ consists of the sets of the form

$$\mathbb{H} = \{U \in \mathcal{O}(Y) : Y \setminus U \in \mathbb{L}\},$$

where $\mathbb{L}$ is a strong Ω-Scott open set of $\mathcal{K}(Y)$.

Obviously, if Ω is the class of all directed sets, then by Theorem 4.1 every strong Ω-Scott open set is strong Scott open.

4.3. Theorem. *Let α and β be as in Theorem 2.1. Then there exists a class Ω of directed sets such that the strong Ω-Scott topology coincides with the strong (α, β)-Scott topology on $\mathcal{K}(Y)$.*

The proof of this theorem is similar to the proof of Theorem 2.1.

4.4. Theorem. *Let B be an Ω-bounded subset of Y. Then the set*

$$\mathbb{H} \equiv \{U \in \mathcal{O}(Y) : B \subseteq U\}.$$

is strong Ω-Scott open in $\mathcal{O}(Y)$.

The proof of this theorem is similar to the proof of Theorem 3.3.

II. Ω-splitting and Ω-jointly continuous topologies

In this section we define some topologies on the set $C(Y, Z)$ using the "Ω-notions" defined in Section I and prove some relations between them.

By Y and Z we denote two fixed topological spaces. If τ is a topology on the set $C(Y, Z)$, then the corresponding topological space is denoted by $C_\tau(Y, Z)$.

Let X be a space and $F : X \times Y \to Z$ be a continuous map. By F_x, where $x \in X$, we denote the continuous map of Y into Z for which $F_x(y) = F(x, y)$ for every $y \in Y$. By $\hat{F}$ we denote the map of X into the set $C(Y, Z)$ for which $\hat{F}(x) = F_x$ for every $x \in X$.

Let G be a map of the space X into the set $C(Y, Z)$. By $\tilde{G}$ we denote the map of the space $X \times Y$ into the space Z for which $\tilde{G}(x, y) = G(x)(y)$ for every $(x, y) \in X \times Y$. It is easy to verify that $\hat{\tilde{G}} = G$ and $\tilde{\hat{F}} = F$.

We recall that a topology τ on $C(Y, Z)$ is called (see [A-D]) *splitting* (respectively, *jointly continuous*) if for every space X, the continuity of a map $F : X \times Y \to Z$ (respectively, of a map $G : X \to C_\tau(Y, Z)$) implies that of the map $\hat{F} : X \to C_\tau(Y, Z)$ (respectively, of the map $\tilde{G} : X \times Y \to Z$).

If in the above definitions of splitting and jointly continuous topologies the space X belongs to a given class $\mathcal{A}$ of spaces, then we have the notions of $\mathcal{A}$-splitting and A-jointly continuous topologies. (See [G-I-P]).

For every $\Lambda \in \Omega$ we consider the set $\mathrm{Sp}(\Lambda) \equiv \Lambda \cup \{\infty\}$, where ∞ is a symbol such that $\lambda \neq \infty$, for every $\lambda \in \Lambda$. On the set $\mathrm{Sp}(\Lambda)$ we define a topology as follows (see [A-D]): a subset U of $\mathrm{Sp}(\Lambda)$ is open if and only if either $\infty \notin U$ or $\infty \in U$ and there exists an element $\lambda_0 \in \Lambda$ such that $\lambda \in U$ for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. By $\mathrm{Sp}(\Omega)$ we denote the family of all spaces $\mathrm{Sp}(\Lambda)$, where $\Lambda \in \Omega$. Instead of "$\mathrm{Sp}(\Omega)$-splitting" or "$\mathrm{Sp}(\Omega)$-jointly continuous" we write "Ω-splitting" or "Ω-jointly continuous", respectively.

By $\mathcal{C}^*$ we denote the class of all pairs $(\{f_\lambda, \lambda \in \Lambda\}, f)$ where $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $C(Y, Z)$, which converges continuously to $f \in C(Y, Z)$ (see [F] and [K]). If τ is a topology on $C(Y, Z)$, then by $\mathcal{C}(\tau)$ we denote the class of all pairs $(\{f_\lambda, \lambda \in \Lambda\}, f)$, where $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $C(Y, Z)$, which topologically converges to $f \in C(Y, Z)$. By $\mathcal{C}^*_\Omega$ (respectively, by $(\mathcal{C}(\tau))_\Omega$) we denote the subclass of all elements $(\{f_\lambda, \lambda \in \Lambda\}, f)$ of C^* (respectively, of $(\mathcal{C}(\tau))$ for which $\Lambda \in \Omega$.

For every set X and for every class $\mathcal{C}$ of pairs $(\{x_\lambda, \lambda \in \Lambda\}, x)$, where $x \in X$ and $\{x_\lambda, \lambda \in \Lambda\}$ is a net in X we denote by $\tau(\mathcal{C})$ the topology on X such that $U \in \tau(\mathcal{C})$ if and only if for every element $(\{x_\lambda, \lambda \in \lambda\}, x) \in \mathcal{C}$ where $x \in U$ there exists an element $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$.

We recall now the chraracterization of continuous convergence on $C(Y, Z)$ which was given in [I-P]. A net $\{f_\lambda, \lambda \in \Lambda\}$ in $C(Y, Z)$ converges continuously to $f \in C(Y, Z)$ if and only if the following condition holds:

$$\overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \subseteq f^{-1}(K),$$

for every closed subset K of Z.

The following criteria are given in [A-D]:

(1) A topology τ on $C(Y, Z)$ is splitting if and only if $\mathcal{C}^* \subseteq \mathcal{C}(\tau)$.
(2) A topology τ on $C(Y, Z)$ is jointly continuous if and only if $\mathcal{C}(\tau) \subseteq \mathcal{C}^*$.

We observe that from the above criteria it follows that there exists at most one topology on $C(Y, Z)$, which is simultaneously splitting and jointly continuous.

The next theorem shows that analogous criteria are true for the Ω-splitting and Ω-jointly continuous topologies on $C(Y, Z)$. The ineterest of this theorem relies on the fact that, in general, there exist a lot of topologies on $C(Y, Z)$, which are simultaneously Ω-splitting and Ω-jointly continuous.

1. Theorem. *The following are true:*

(1) A topology τ on $C(Y,Z)$ is Ω-splitting if and only if

$$\mathcal{C}^*_\Omega \subseteq (\mathcal{C}(\tau))_\Omega.$$

(2) A topology τ on $C(Y,Z)$ is Ω-jointly continuous if and only if $(\mathcal{C}(\tau))_\Omega \subseteq C^*_\Omega$.

(3) The topology $\tau(\mathcal{C}^*_\Omega)$ is the greatest Ω-splitting topology on $C(Y,Z)$ (see [G-I-P]), that is,

$$\tau(\mathcal{C}^*_\Omega) = \tau(Sp(\Omega)).$$

Proof. The proofs of the statements (1) and (2) of the theorem are similar to the proofs of the corresponding results (criteria (1) and (2)) of [A-D].

(3). Let $\tau = \tau(\mathcal{C}^*_\Omega)$. By the definition of the topology $\tau(\mathcal{C}^*_\Omega)$, $\mathcal{C}^*_\Omega \subseteq (\mathcal{C}(\tau))_\Omega$. By statement (1), the topology τ is an Ω-splitting topology on $C(Y,Z)$. Let τ_1 be an Ω-splitting topology on $C(Y,Z)$. By (1) we have $\mathcal{C}^*_\Omega \subseteq (\mathcal{C}(\tau_1))_\Omega \subseteq \mathcal{C}(\tau_1)$. Hence $\tau(\mathcal{C}^*_\Omega) \subseteq \tau(\mathcal{C}(\tau_1)) = \tau_1$. Thus $\tau(\mathcal{C}^*_\Omega)$ is the greatest Ω-splitting topology, that is, $\tau(\mathcal{C}^*_\Omega) = \tau(\mathrm{Sp}(\Omega))$.

2. Definitions. As it is well-known the topology τ_{co} on $C(Y,Z)$ for which the sets

$$(K,U) = \{f \in C(Y,Z) : f(K) \subseteq U\},$$

compose a subbasis of open sets, where K is an compact subset of Y and U is an open subset of Z, is called *compact-open topology* on $C(Y,Z)$. If in the above definition instead of compact we consider as Ω-compact subsets, we get the Ω-compact open topology on $C(Y,Z)$ which is denoted by $\tau^\Omega_{\mathrm{co}}$.

The topology τ_{is} on $C(Y,Z)$ for which the sets

$$(\mathbb{H},U) = \{f \in C(Y,Z) : f^{-1}(U) \in \mathbb{H}\}$$

compose a subbasis of open sets, where $\mathbb{H}$ is a Scott open set of $\mathcal{O}(Y)$ and U is an open subset of Z, is called *Isbell topology* on $C(Y,Z)$. If in the above definition instead of Scott open we consider Ω-Scott open sets, we get the Ω-Isbell topology on $C(Y,Z)$ which is denoted by $\tau^\Omega_{\mathrm{is}}$.

We observe that if $\mathbb{H}$ is an Ω-Scott open set of $\mathcal{O}(Y)$, $\mathbb{L} = \{\mathcal{K}(Y) : Y \setminus F \in \mathbb{H}\}$, $U \in \mathcal{O}(Y)$ and $K = Y \setminus U$, then

$$(\mathbb{H},U) = (\mathbb{L},K) = \{f \in C(Y,Z) : f^{-1}(K) \in \mathbb{L}\}.$$

3. Theorem. *The Ω-Isbell topology on $C(Y,Z)$ is Ω-splitting.*

Proof. By Theorem 1, it is sufficient to prove that $\mathcal{C}^*_\Omega \subseteq (\mathcal{C}(\tau))_\Omega$. Let $\{f_\lambda, \lambda \in \Lambda\}$ be an Ω-net in $C(Y,Z)$ and $f \in C(Y,Z)$ such that $\overline{\lim}_\Lambda(f_\lambda^{-1}(K)) \subseteq f^{-1}(K)$ for each closed subset K of Z, that is $(\{f_\lambda, \lambda \in \Lambda\}, f) \in \mathcal{C}^*_\Omega$. We prove that the Ω-net $\{f_\lambda, \lambda \in \Lambda\}$ converges to f with respect to the Ω-Isbell topology. Let $f \in (\mathbb{L}, K)$, where $\mathbb{L}$ is a Scott open set of $\mathcal{K}(Y)$ and K is closed in Z. Then $f^{-1}(K) \in \mathbb{L}$ and hence $\overline{\lim}_\Lambda(f_\lambda^{-1}(K)) \in \mathbb{L}$. So there exists a $\lambda_0 \in \Lambda$ such that for every $\lambda \geq \lambda_0$, $f_\lambda^{-1}(K) \in \mathbb{L}$, which means that the Ω-net $\{f_\lambda : \lambda \in \Lambda\}$ converges to f witht respect to the Ω-Isbell topology.

4. Theorem. *The topology $\tau^\Omega_{\mathrm{co}}$ is contained in the topology $\tau^\Omega_{\mathrm{is}}$.*

Proof. It is sufficient to observe that $(K,U) = (\mathbb{H}, U)$ for every Ω-compact subset K of Y and for every open subset U of Z, where $\mathbb{H} = \{U \in \mathcal{O}(Y) : K \subseteq U\}$.

5. Corollary. *The Ω-compact-open topology on $C(Y,Z)$ is Ω-splitting.*

6. Definition. A space X is called *locally Ω-compact* if for every $x \in X$ and for every open neighbourhood V containing x there exists an Ω-compact neighbourhood U of x in X such that $x \in U \subseteq V$.

7. Theorem. *If Y is a locally Ω-compact space, then the Ω-compact open topology on $C(Y,Z)$ is Ω-jointly continuous.*

Proof. Let G be a continuous map of a space $X \equiv \mathrm{Sp}(\Lambda)$ into the space $C_\tau(Y,Z)$, where $\Lambda \in \Omega$ and $\tau \equiv \tau^\Omega_{\mathrm{co}}$. We prove that the map $\tilde{G}$ of $X \times Y$ into Z is continuous. It is sufficient to prove that $\tilde{G}$ is continuous at the points (∞, y) of $X \times Y$.

Let $(\infty, y) \in X \times Y$ and U be an open neighbourhood of $\tilde{G}(\infty, y) = G(\infty)(y)$ in Z. Since Y is locally Ω-compact and the map $G(\infty)$ is continuos there exists an Ω-compact neighbourhood of K of y in Y such that $K \subseteq (G(\infty))^{-1}(U)$. Hence the set (K,U) is an open neighbourhood of $G(\infty)$ in $C_\tau(Y,Z)$. Since G is continuous there exists an open neighbourhood W of ∞ in X such that $G(W) \subseteq (K,U)$. There exists an element $\lambda_0 \in \Lambda$ such that $\lambda \in W$ for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Hence $G(\lambda) \in (K,U)$, that is, $G(\lambda)(K) \subseteq U$ for every $\lambda \geq \lambda_0$.

The subset $W \times V$ of $X \times Y$ where $V \subseteq K$, is an open neighbourhood of y in Y, is an open neighbourhood of (∞, y) in $X \times Y$. Let $(x, y') \in W \times V$. Then $\tilde{G}(x,y') = G(x)(y') \in U$, that is, $\tilde{G}(W \times V) \subset U$. Hence the map $\tilde{G}$ is continuous. Thus, the topology $\tau^\Omega_{\mathrm{co}}$ is Ω-jointly continuous.

8. Theorem. *A space Y is corecompact if and only if for every net $\{K_\lambda, \lambda \in \Lambda\}$ in $\mathcal{K}(Y)$, which converges in the Scott topology to a closed set K of Y, we have $\overline{\lim}_\Lambda(K_\lambda) \subseteq K$.*

PROOF. The proof of this theorem immediately follows by the fact that the space Y is corecompact if and only if the Isbell topology on the set $C(Y, \mathbf{S})$ is jointly continuous. (See [L-P]).

Using the above theorem we give the following definition.

9. Definition. A space Y is Ω-corecompact if for every Ω-net $\{K_\lambda, \lambda \in \Lambda\}$ in $\mathcal{K}(Y)$, which converges in the Scott topology to a closed set K of Y, we have $\overline{\lim}_\Lambda(K_\lambda) \subseteq K$.

10. Theorem. *Let Y be an Ω-corecompact space. Then the Ω-Isbell topology on $C(Y, Z)$ is Ω-jointly continuous.*

PROOF. It is sufficient to prove that $(\mathcal{C}(\tau))_\Omega \subseteq \mathcal{C}^*_\Omega$. Let $\{f_\lambda, \lambda \in \Lambda\}$ be an Ω-net, which converges to $f \in C(Y, Z)$ with respect to the Ω-Isbell topology. We prove that $(\{f_\lambda, \lambda \in \Lambda\}, f) \in \mathcal{C}^*_\Omega$, that is, $\overline{\lim}_\Lambda(f_\lambda^{-1}(K)) \subseteq f^{-1}(K)$ for each closed K of Z. Obviously for each closed set K of Z the Ω-net $\{f_\lambda^{-1}(K), \lambda \in \Lambda\}$ converges to $f^{-1}(K)$ with respect to the Ω-Scott topology. Since Y is corecompact, by Theorem 8, $\overline{\lim}_\Lambda(f_\lambda^{-1}(K)) \subseteq f^{-1}(K)$. Thus the Ω-Isbell topology is Ω-jointly continuous.

11. Definition. The topology τ_b^Ω on $C(Y, Z)$ for which the sets

$$(B, U) = \{f \in C(Y, Z) : f(B) \subseteq U\}$$

compose a subbasis of open sets, where B is an Ω-bounded subset of Y and U is an open subset of Z, is called Ω-*bounded topology*. (For the *bounded* topology τ_b on $C(Y, Z)$ see [L_2]).

12. Theorem. *If Y is locally Ω-compact, then the topology τ_b^Ω on $C(Y, Z)$ is Ω-jointly continuous.*

PROOF. By property (5) of Section 3.2, we have that $\tau_{\mathrm{co}}^\Omega \subseteq \tau_b^\Omega$. Since the space Y is locally Ω-compact by Theorem 7 the topology $\tau_{\mathrm{co}}^\Omega$ is Ω-jointly continuous and hence τ_b^Ω is also Ω-jointly continuous.

13. Definition. A space X is called *locally* Ω-bounded if every $x \in X$ has an Ω-bounded neighbourhood U in X.

14. Theorem. *If Y is locally Ω-bounded, then the Ω-bounded topology on $C(Y, Z)$ is Ω-jointly continuous.*

The proof of this theorem is similar to the proof of Theorem 7.

15. Definition. The topology on $C(Y,Z)$ for which the sets

$$(\mathbb{H}, U) = \{f \in C(Y,Z) : f^{-1}(U) \in \mathbb{H}\}$$

compose a subbasis of open sets, where $\mathbb{H}$ is a strong Ω-Scott open set of $\mathcal{O}(Y)$, is called *strong Ω-Isbell topology* on $C(Y,Z)$. This topology is denoted by $\tau^{\Omega}_{\text{s-is}}$.

16. Theorem. *If Y is locally Ω-bounded, then the strong Ω-Isbell topology on $C(Y,z)$ is Ω-jointly continuous.*

PROOF. By Theorem 4.4 the topology τ^{Ω}_{b} is contained in the topology $\tau^{\Omega}_{\text{s-is}}$ and hence by Theorem 14, $\tau^{\Omega}_{\text{s-is}}$ is Ω-jointly continuous.

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