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CR-submanifolds of the paracomplex projective space

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Abstract. Some basic facts about CR-submanifolds of almost para-Hermitian manifolds are stated. Examples of CR-submanifolds of paracomplex projective space are given. It is proved that degenerate hypersurfaces are CR-submanifolds if and only if they are invariant, that is, $\bar{J}(T_xM)^{\perp} = (T_xM)^{\perp}$.

1 Introduction

CR-submanifolds of almost Hermitian manifolds have been introduced by Bejancu in the seventies (see [4] for a general treatment). The study of invariants of real hypersurfaces of almost complex manifolds goes back to the work of E. Cartan [6] in 1932, and it has been an active field of research in the past two decades, with relations to complex analysis, partial differential equations, and mathematical physics (see [7] and the bibliography in [13]).

On the other hand, there is the theory of almost para-complex manifolds (see the bibliography in [8]). While the J operator for an almost complex manifold obeys $J^2 = -I$, the coresponding operator for an almost para-complex manifold obeys $J^2 = I$. This para-complex theory bears many similarities with the complex theory and some differences. In particular, almost para-Hermitian manifolds have necessarily a neutral metric, i.e., a pseudo-Riemannian metric of signature (n, n). This adds an extra difficulty to the theory of CR-submanifolds of almost para-Hermitian manifolds. The theory is of local nature.

Some special cases of these submanifolds have been studied by Amato, and Rosca [1, 2, 14, 15, 16]. However, no concrete examples have

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been given up to date. In this work, we provide some examples of CR-submanifolds of the paracomplex projective space $P_n(B)$. This 2n-dimensional space $P_n(B)$ has been introduced by Gadea and Montesinos as the model of para-Kählerian manifold of constant para-holomorphic sectional curvature [11].

In section 2 we present the generalities of the theory of CR-submanifolds of an almost para-Hermitian manifold, showing its relation to the theory of $\phi(4,-2)$-manifolds [10, 17], and degenerate hypersurfaces [5]. In section 3 we present some examples of CR-submanifolds of $P_n(B)$, namely, the sphere S^n (Proposition 4), the integral submanifolds of the eigenspaces of eigenvalues ± 1 of the almost product structure on $P_n(B)$ (Proposition 5), the space $P_k(B)$ with $k < n$ (Proposition 6), and a CR-coisotropic hypersurface of defect one (Proposition 7).

2 CR-submanifolds of an almost para-Hermitian manifold

Let $(\overline{M}, \bar{J}, \bar{g})$ be an almost para-Hermitian manifold, that is, $\overline{M}$ is a real manifold of even dimension $2\overline{m}$, $\bar{J}$ is a tensor field of type (1,1) verifying $\bar{J}^2 = I, \bar{J} \neq \pm I, I$ being the identity, and $\bar{g}$ is a pseudo-Riemannian metric of signature $(\overline{m}, \overline{m})$ such that $\bar{g}(\bar{J}\bar{X}, \bar{Y}) + \bar{g}(\bar{X}, \bar{J}\bar{Y}) = 0$, for all vector fields $\overline{X}, \overline{Y}$ on $\overline{M}$.

Definition 1. A submanifold M of $\overline{M}$ is a *CR-submanifold* of $\overline{M}$ if the following conditions are satisfied:

(a) The metric $\bar{g}|_M = g$ is of constant signature and rank.
(b) There exists two differentiable distributions D and $D^\perp$ on M satisfying:
  (b.1) D is invariant, i.e, $\bar{J} D_x = D_x, \forall x \in M$,
  (b.2) $D^\perp : x \to D_x^\perp \subset T_x M$ is anti-invariant, i.e., $\bar{J}(D_x^\perp) \subset (T_x M)^\perp$,
  (b.3) $T_x M = D_x \oplus D_x^\perp$, $\forall x \in M$, and D_x, $D_x^\perp$ are mutually orthogonal.

Notation. Induced objects on M are denoted by suppressing the bar. The number dim $M -$ rank g will be called the *defect* of g.

Definition 2. Let M be a CR-submanifold of an almost para-Hermitian manifold $(\overline{M}, \bar{J}, \bar{g})$, and let $D, D^\perp$ be the corresponding distributions on M.

(a) M is an *invariant* CR-submanifold if $D^\perp = \{0\}$.
(b) M is an *anti-invariant* CR-submanifold if $D = \{0\}$.

We prefer not to use the term proper in the case $D \neq \{0\} \neq D^\perp$, because the term proper is very often used for a submanifold of a pseudo-Riemannian manifold with non degenerate induced metric.

Remarks. (1) $\dim (T_xM)^\perp + \dim T_xM = \dim T_x\overline{M}$, $\forall x \in M$, but in general, $(T_xM)^\perp \cap T_xM = \Delta_x M$ may be different from zero. In fact, there is a class of CR-submanifolds called co-isotropic, which verify $(T_xM)^\perp \subset T_xM$, $\forall x \in M$ [15].

(2) Nothing is said in the definitions about the dimension of M.

(3) Condition (b.3) in definition 1 is not superfluous, since g might be degenerate.

We shall develope the theory of CR-submanifolds in analogy with the theory of CR-submanifolds of almost Hermitian manifolds given in [4]. As we have the decomposition $T_xM = D_x \oplus D_x^\perp$ we can define the projections $P_x : T_xM \longrightarrow D_x$ and $Q_x : T_xM \longrightarrow D_x^\perp$. Let ϕ_x, ω_x be the following maps

$$\phi_x = \bar{J}_x \circ P_x\, ,\ \omega_x = \bar{J}_x \circ Q_x .$$

Lemma. *With the above notations, $\phi^2 = P$.*

Proof. Let $X \in T_xM$. Then, $\phi_x^2(X) = \bar{J}_x P_x \bar{J}_x P_x(X) = \bar{J}_x \bar{J}_x P_x(X) = P_x(X)$, because $\bar{J}_x P_x(X) \in D_x$ and $\bar{J}_x^2 = I$.

As a consequence of the lemma, ϕ is a tensor field of type (1,1) on M verifying $\phi^4 = \phi^2$. A manifold M of even dimension, endowed with a tensor field of type (1,1) satisfying $\phi^4 - \phi^2 = 0$, rank $\phi = \frac{1}{2}(\text{rank } \phi^2 + \dim M)$, is called a $\phi(4, -2)$-manifold [10, 17]. In our case, the condition on the rank of ϕ is never satisfied, and the dimension of M may be odd. Nevertheless, we can obtain some results on the integrability of the distributions D and $D^\perp$, using some theorems on $\phi(4, -2)$-manifolds.

Let $\ell = \phi^2 = P$, $\bar{\ell} = I - \phi^2 = I - P = Q$. Then $D = \text{Im } P = \text{Im } \ell$, $D^\perp = \text{Im } Q = \text{Im } \bar{\ell}$. Using a result of [10] on the integrability of the distributions Im ℓ, Im $\bar{\ell}$ we get

Proposition 1. *Let N_ϕ be the Nijenhuis tensor of ϕ . With the above assumptions:*

(1) D is involutive iff $QN_\phi(PX, PY) = 0$, $\forall X, Y \in \Gamma(TM)$;

(2) $D^\perp$ is involutive iff $PN_\phi(QX, QY) = 0$, $\forall X, Y \in \Gamma(TM)$.

Example 1. Let us assume that M is a proper pseudo-Riemannian submanifold of $\overline{M}$, i.e., that the immersion $M \to \overline{M}$ is proper or, equivalently, $g \equiv \bar{g}|_M$ is a non-degenerate metric. Then, the theory is close to that of CR-submanifolds of an almost Hermitian manifold. For example, if $X \in \Gamma(TM)$, then $\bar{J}X = \bar{J}PX + \bar{J}QX = \phi X + \omega X$, where

$\phi X \in \Gamma(TM)$, $\omega X \in \Gamma(TM)^\perp$. In addition, $T_x\overline{M} = T_xM \oplus (T_xM)^\perp$, and this decomposition allows to define ϕ (resp. ω) as the first (resp. the second) projection. This is the starting point in [4]. Also, Gauss and Weingarten equations (following the general theory of pseudo-Riemannian submanifolds), and some examples, are shown in [3].

Example 2. Degenerate hypersurfaces of almost para-Hermitian manifolds are studied in [5]. We would like to show their relations to the theory of CR-submanifolds. A real hypersurface M of an almost para-Hermitian manifold $(\overline{M}, \bar{J}, \bar{g})$ is called a *degenerate hypersurface* if

$$\Delta : x \longrightarrow \Delta_x = (T_xM)^\perp \cap T_xM$$

defines a non-trivial distribution on M. In this case, $(T_xM)^\perp$ has dimension one, $(T_xM)^\perp \subset T_xM$, and $(T_xM)^\perp$ is isotropic. Moreover $g \equiv \bar{g}|_M$ verifies rank $g = \dim M - 1$. There are two classes of degenerate hypersurfaces. *Non-invariant degenerate hypersurfaces* obey $\bar{J}(T_xM)^\perp \cap (T_xM)^\perp = \{0\}$ and M is not invariant with respect to $\bar{J}$. *Invariant degenerate hypersurfaces* obey $\bar{J}(T_xM)^\perp = (T_xM)^\perp$ and M is invariant with respect to $\bar{J}$.

We have the following:

Proposition 2. *An invariant degenerate hypersurface M is a coisotropic CR-submanifold of defect 1.*

Proof. Let $D_x^\perp = (T_xM)^\perp$, and D_x any complement of $D_x^\perp$, that is, $T_xM = D_x \oplus D_x^\perp$. Since M is $\bar{J}$-invariant one can choose D_x so $\bar{J}(D_x) = D_x$, taking into account that $D_x^\perp$ is isotropic, and contained in an eigenspace of $\bar{J}|_M$ (because eigenspaces are maximally isotropic).

Observe that $g|_{D^\perp} = 0$, and $g|_D$ is a non-degenerate metric.

Necessary and sufficient conditions for the integrability of D are given in Theorem 3 of [5], in terms of certain bilinear form associated to a particular distribution orthogonal to $TM^\perp$ and containing $\bar{J}(TM^\perp)$.

The proposition above fails for non-invariant degenerate hypersurfaces. In fact, we have the following

Proposition 3. *A non-invariant degenerate hypersurface M of defect 1 it is not a CR-submanifold.*

Proof. Suppose M is a CR-submanifold with distribution D as in definition 1. Let $X \in (T_xM)^\perp$, $X \neq 0$, and $Y \in T_xM$. By condition (b.3) in Definition 1

$$Y = Y_1 + Y_2 \quad Y_1 \in D_x\,,\ Y_2 \in D_x^\perp$$

then

$$\bar{J}Y = \bar{J}Y_1 + \bar{J}Y_2\,,\ \ \bar{J}Y_1 \in D_x \subset T_xM\,,\ \ \bar{J}Y_2 \in \bar{J}(D_x^{\perp}) \subset (T_xM)^{\perp} \subset T_xM$$

Hence

$$0 = g(\bar{J}X, Y) + g(X, \bar{J}Y) = g(\bar{J}X, Y)$$

whence $\bar{J}X \in (T_xM)^{\perp}$. On the other hand, $\bar{J}X \in \bar{J}(T_xM)^{\perp}$. Thus

$$\bar{J}(T_xM)^{\perp} \cap (T_xM)^{\perp} \neq \{0\},$$

which contradicts the non-invariance of M.

Example 3. A certain class of codimension 2 CR-submanifolds are studied in [15] and [2]. In this case, the metric $g = \bar{g}|_M$ is degenerate and rank $g = \dim\, M - 2$. In addition, the distributions D and $D^{\perp}$ considered by Rosca and Amato verify the following conditions: $D_x^{\perp} = (T_xM)^{\perp} \subset T_xM$ whence M is a coisotropic CR-submanifold; $D_x^{\perp}$ is a 2-dimensional isotropic space, whence $g|_{D^{\perp}} = 0$ and $g|_D$ is a non-degenerate metric. One can check that these conditions are similar to those obtained in Example 2.

Example 4. In [16], the following theorem is proved: *Any coisotropic submanifold of a para-Kählerian manifold is a CR-submanifold.* The proof uses, as above, the distributions

$$D_x^{\perp} = (T_xM)^{\perp} \subset T_xM,$$

which is isotropic, and D a complementary distribution of $D^{\perp}$ in TM. Then $g|_{D^{\perp}} = 0$, and $g|_D$ is non-degenerate.

Example 5. Codimension $\overline{m} - 1$ CR-submanifolds of para-Kählerian manifolds having the Poisson property, are studied in [1]. Amato shows that for such CR-submanifolds both D and $D^{\perp}$ are integrable. Moreover $D^{\perp}$ is isotropic and totally geodesic.

3 CR-submanifolds of the paracomplex projective space

As mentioned in the introduction, the paracomplex projective space $P_n(B)$ was introduced by Gadea and Montesinos as the model of para-Kählerian manifold of constant para-holomorphic sectional curvature [11]. Some geometric properties of this space form have been obtained in [12] and [9]. We mention some of the basic facts of paracomplex projective space, which will be used in the sequel:

(1) $\dim P_n(B) = 2n$;

$$P_n(B) = \{(u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \langle u, u\rangle = \langle v, v\rangle, \langle u, v\rangle = 1\},$$

where $\langle\, ,\rangle$ denotes the standard inner product.

(2) Local charts $(U_\alpha^+, \psi_\alpha)$ and $(U_\alpha^-, \psi_\alpha)$ are defined on $P_n(B)$, where

$$\begin{aligned} U_\alpha^+ &= \{(u,v) \in P_n(B) : u^\alpha > 0, v^\alpha > 0\}, \\ U_\alpha^- &= \{(u,v) \in P_n(B) : u^\alpha < 0, v^\alpha < 0\}, \\ \psi_\alpha(u,v) &= \Big(\frac{u^0}{u^\alpha}, \dots, \frac{\widehat{u^\alpha}}{u^\alpha}, \dots, \frac{u^n}{u^\alpha}; \frac{v^0}{v^\alpha}, \dots, \frac{\widehat{v^\alpha}}{v^\alpha}, \dots, \frac{v^n}{v^\alpha}\Big\} \end{aligned}$$

with $\widehat{}$ denoting a deleted element. The local coordinates are

$$(x^i = \frac{u^i}{u^\alpha}, y^i = \frac{v^i}{v^\alpha}).$$

(3) The para-Kählerian structure is given, in local coordinates, by

$$J = \frac{\partial}{\partial x^i} \otimes dx^i - \frac{\partial}{\partial y^i} \otimes dy^i$$

$$g = \sum_{i,j} \frac{2}{c(1+\langle x,y\rangle)} \Big[dx^i \otimes dy^i + dy^i \otimes dx^i - \frac{1}{1+\langle x,y\rangle} x^i y^j (dy^i \otimes dx^j + dx^j \otimes dy^i) \Big]$$

so $(P_n(B), J, g)$ is a para-Kählerian manifold of constant paraholomorphic sectional curvature c.

(4) There exists a global diffeomorphism

$$\varphi_n : P_n(B) \longrightarrow TS^n, (u,v) \mapsto \left(\frac{u+v}{\|u+v\|}, u-v \right).$$

Then we obtain the following results:

Proposition 4. *There exists a canonical embedding $i_n : S^n \longrightarrow P_n(B)$ which makes of S^n an anti-invariant CR-submanifold of $P_n(B)$. This embedding, composed with φ_n , gives the canonical embedding of S^n into TS^n, as the null section.*

Proof. Let $S^n = \{u \in \mathbb{R}^{n+1}, \sum_{i=0}^{n} (u^i)^2 = 1\}$, and define $i_n(u) = (u,u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Obviously, $i_n(u) \in P_n(B)$, and $i_n : S^n \longrightarrow P_n(B)$ is an embedding. One can check that $\varphi_n \circ i_n$ transforms S^n onto the null section of TS^n. From the definition of S^n, if $u \in S^n$, there exists $\alpha \in \{0, 1, \dots, n\}$ such that $u^\alpha \neq 0$. Then, we work with the corresponding chart $(U_\alpha^\pm, \psi_\alpha)$ and local coordinates (x^i, y^i). A simple calculation shows that $i_n(S^n)$ is given by local equations $\{x^i = y^i\}$, and $T_{(u,u)}(i_n(S^n))$ is generated by

$$\left\{ \left(\frac{\partial}{\partial x^i}\right)_u + \left(\frac{\partial}{\partial y^i}\right)_u \right\},$$

verifying

$$J\left(\left(\frac{\partial}{\partial x^i}\right)_u + \left(\frac{\partial}{\partial y^i}\right)_u\right) = \left(\frac{\partial}{\partial x^i}\right)_u - \left(\frac{\partial}{\partial y^i}\right)_u,$$

$$\text{and} \qquad g\left(\left(\frac{\partial}{\partial x^i}\right)_u + \left(\frac{\partial}{\partial y^i}\right)_u, \left(\frac{\partial}{\partial x^j}\right)_u - \left(\frac{\partial}{\partial y^j}\right)_u\right) = 0\,,$$

thus proving the result.

A direct consequence of the definition is the following:

Lemma. *If (M, J) is an almost product structure, i.e., J is a tensor field of type $(1,1)$ verifying $J^2 = I$, and if S_+ (resp. S_-) denotes the distribution corresponding to the eigenvalue 1 (resp. -1), then S_+ and S_- are involutive if and only if $N_J = 0$, N being the Nijenhuis tensor of J.*

We apply this lemma to $M = P_n(B)$. Since $P_n(B)$ is a para-Kählerian manifold, $N_J = 0$, and then we obtain

Proposition 5. *Any integral manifold N of S_+ or S_- is an invariant CR-submanifold of $P_n(B)$.*

Proof. If N is an integral manifold of S_+ or S_-, then $J|_N = \pm I$. Taking $D_p = T_pN$, $D_p^\perp = 0$ for each $p \in N$, the result follows.

Note that the induced metric on N is null, and so it is degenerate.

Proposition 6. *For all $k < n$, there exists an isometric embedding $j_{k,n} : P_k(B) \longrightarrow P_n(B)$ as an invariant CR-submanifold.*

Proof. Consider the injection $\mathbb{R}^{k+1} \longrightarrow \mathbb{R}^{n+1}$ given by

$$u = (u^0, \dots, u^k) \mapsto \bar{u} = (u^0, \dots, u^k, 0, \dots, 0),$$

and obtain the induced injection

$$j_{k,n} : P_k(B) \longrightarrow P_n(B), \quad (u, v) \mapsto (\bar{u}, \bar{v}).$$

This is an embedding. In the local charts $(U_\alpha^\pm, \psi_\alpha)$ and $(\bar{U}_\alpha^\pm, \bar{\psi}_\alpha)$ on $P_n(B)$, $j_{k,n}$ is given by

$$(x^1, \dots, x^k; y^1, \dots, y^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0),$$

and then $j_{k,n}(P_k(B)) = \{x^{k+1} = \cdots = x^n = y^{k+1} = \cdots = y^n = 0\}$. Thus, $P_k(B)$ is J-invariant in $P_n(B)$.

The examples of CR-submanifolds of $P_n(B)$ given in Propositions 3, 4 and 5 are either invariant or anti-invariant. We shall show now a CR-submanifold such that $D \neq \{0\}$, $D^\perp \neq \{0\}$. For the sake of simplicity, we work with $n = 2$. Then, in a local chart $(U_\alpha^\pm; x^1, x^2, y^1, y^2)$ the metric g has the following matrix expression:

$$g = \frac{2}{c\lambda} \begin{pmatrix} 0 & 0 & 1 - \frac{x^1y^1}{\lambda} & -\frac{x^2y^1}{\lambda} \\ 0 & 0 & -\frac{x^1y^2}{\lambda} & 1 - \frac{x^2y^2}{\lambda} \\ 1 - \frac{x^1y^1}{\lambda} & -\frac{x^1y^2}{\lambda} & 0 & 0 \\ -\frac{x^2y^1}{\lambda} & 1 - \frac{x^2y^2}{\lambda} & 0 & 0 \end{pmatrix}$$

where $\lambda = 1 + \langle x, y\rangle$.

Proposition 7. *The hypersurface $S = \{u^2 = 0\} \subset P_2(B)$ is a coisotropic CR-submanifold of defect one.*

Remarks. (1) The hypersurface above is an invariant degenerate hypersurface in the sense of [5] (see also Proposition 2).

(2) Proposition 7 gives an example of the following theorem in [16]: *Any coisotropic submanifold of a para-Kählerian manifold is a CR-submanifold with involutive vertical distribution $D^\perp$, and the leaves of $D^\perp$ are isotropic.*

Proof of Proposition 7. First, observe that the hypersurface is locally defined by the equation $\{x^2 = 0\}$, in the charts $(U_0^\pm, \psi_0)$ and $(U_1^\pm, \psi_1)$. Let us fix a local chart $(U_\alpha^\pm, \psi_\alpha)$, $\alpha = 0, 1$, with local coordinates (x^1, x^2, y^1, y^2). The tangent space to S in each point is generated by $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right\}$. We shall determine the rank of $g|_S$ by computing the metric coefficients with respect to this local basis:

$$g|_S = \frac{2}{c(1 + \langle x, y\rangle)} \begin{pmatrix} 0 & A & 0 \\ A & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $A = 1 - \dfrac{x^1y^1}{1 + \langle x, y\rangle} \neq 0$. Then, rank $(g|_S) = 2$, and the line generated by $\dfrac{\partial}{\partial y^2}$ is isotropic. Furthermore $D^\perp = \left\langle \dfrac{\partial}{\partial y^2} \right\rangle$ is orthogonal to any transversal distribution D. We can choose $D = \left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1} \right\rangle$. This D defines the CR-submanifold structure of S. Observe that

$$J(D_x) = D_x, J(D_x^\perp) = (T_xS)^\perp = D_x^\perp, \quad \forall x \in S,$$

whence S is a coisotropic submanifold.

This proposition becomes generalized to higher dimensions in a straightforward way.

Remark. With the notation of Proposition 6, the composition

$$\beta : S \hookrightarrow P_2(B) \xrightarrow{\psi_2} TS^2 \to S^2$$

is not surjective. In fact, the point $(0,0,1) \in S^2$ has no pre-image. If $\beta(u^0, u^1, 0; v^0, v^1, v^2) = (0,0,1)$, then $u^0 + v^0 = 0$, $u^1 + v^1 = 0$, $v^2 \neq 0$, which is impossible, because $(u^0)^2 + (u^1)^2 = (v^0)^2 + (v^1)^2 + (v^2)^2$.

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