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Families of mappings and fixed points

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Abstract. In this paper, some necessary and sufficient conditions for the existence of fixed points of a family of self-mappings of a metric space are given and a fixed point theorem for a compact mapping is established.

1. Introduction

JUNGCK [1] first gave a necessary and sufficient condition for the existence of fixed points of a continuous self-mapping of a complete metric space. Afterwards, PARK [2] and KHAN and FISHER [3] established a few theorems similar to that of Jungck. JANOS [4] and PARK [5] proved fixed point theorems for compact self-mappings of a metric space.

The purpose of this paper is to offer some characterizations for the existence of fixed points of a family of self-mapings of metric spaces. We also obtain a fixed point theorem for a compact mapping, which extends properly the results of JANOS [4] and PARK [5].

ω and $\mathbb{N}$ denote the sets of nonnegative and positive integers, respectively. Let f be a self-mapping of a metric space (X, d). Following FURI and VIGNOLI [6], f is said to be condensing if for every bounded subset A of X with $\alpha(A) > 0$, we have $\alpha(fA) < \alpha(A)$, where $\alpha(A)$ denotes the measure of noncompactness in the sense of Kuratowski. Define $C_f = \{g \mid g{:}X \to X$ and $fg = gf\}$, $H_f = \{g \mid g : X \to X$ and $g \bigcap_{n\in\omega} f^n X \subset \bigcap_{n\in\omega} f^n X\}$, $O(x, f) = \{f^n x \mid n \in \omega\}$ and $O(x, y, f) = O(x, f) \cup O(y, f)$ for $x, y \in X$. Let $\mathcal{F}$ be a family of self-mappings of X. A point $x \in X$ is said to be a fixed point of $\mathcal{F}$ if $fx = x$ for all $f \in \mathcal{F}$. Set $\mathcal{F} = \{F \mid F$ is a real-valued lower semi-continuous function of $X \times X$ into $[0, \infty)$ such that $F(x, y) = 0$

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if and only if $x = y\}$. For $A \subset X$, $\delta(A)$ and $\overline{A}$ denote the diameter and closure of A, respectively. Let $M(X)$ denote the set of all metrics on X that are topologically equivalent to d for a given metric space (X, d).

Remark. Clearly, $H_f \supset C_f \supset \{f^n \mid n \in \omega\}$.

2. Fixed point theorems

Theorem 1. *Let $\mathcal{F}$ be a family of self-mappings of a bounded metric space (X,d). Then $\mathcal{F}$ has a fixed point if and only if there exists a continuous compact self-mapping f of X such that $f \in \bigcap_{g \in \mathcal{F}} C_g$ and*

$$(*) \qquad d(fx, fy) < \delta\left(\bigcup_{h \in H_f} O(x, y, h)\right) \text{ for all } x, y \in X \text{ with } x \neq y.$$

Proof. To see that the stated condition is necessary, suppose that $\mathcal{F}$ has a fixed point $w \in X$. Define a mapping $f : X \to X$ by $fx = w$ for all $x \in X$. Then $fgx = w = gw = gfx$ for all $g \in \mathcal{F}$ and all $x \in X$; i.e., $f \in \bigcap_{g \in \mathcal{F}} C_g$. Clearly, $(*)$ holds.

On the other hand, suppose there exists a continuous compact self-mapping f of X such that $f \in \bigcap_{g \in \mathcal{F}} C_g$ and $(*)$ hold. Since f is compact, there exists a compact set Y with $fX \subset Y \subset X$. Consequently, $X \supset Y \supset \ldots \supset f^n X \supset f^n Y \supset f^{n+1} X \supset f^{n+1} Y \supset \ldots$ for $n \in \omega$. Set $A = \bigcap_{n \in \omega} f^n X$ and $B = \bigcap_{n \in \omega} f^n Y$. Note that $A = \bigcap_{n \in \mathbb{N}} f^n X \subset B \subset A$. It follows that $A = B$. Since f is continuous and Y is compact and $f^{n+1} Y \subset f^n Y$ for $n \in \omega$, it follows that B is a nonempty compact set and $fB \subset \bigcap_{n \in \omega} f^{n+1} Y \subset B$. We now show that $fB \supset B$. Given $b \in B = \bigcap_{n \in \omega} f^n Y$, there exists $x_n \in f^n Y$ with $b = fx_n$ for $n \in \mathbb{N}$. Note that $\{x_n\}_{n \in \mathbb{N}} \subset Y$. Hence we can extract a subsequence $\{x_{n_i}\}_{i \in \mathbb{N}}$ converging to $p \in Y$. For every $m \in \mathbb{N}$, there exists $i > m$ such that $\{x_{n_i}, x_{n_{i+1}}, x_{n_{i+2}}, \ldots\} \subset f^m Y$. Compactness of $f^m Y$ implies $x_{n_i} \to p \in f^m Y$ as $i \to \infty$. Therefore $p \in \bigcap_{n \in \mathbb{N}} f^n Y = B$. By the continuity of f, we have $b = fx_{n_i} \to fp$ as $i \to \infty$; i.e., $b = fp \in fB$. This proves $fB \supset B$. Hence $fB = B$. Thus A is a nonempty compact set and $fA = A$. Consequently, we can find $x, y, u, v \in A$ with $\delta(A) = d(u, v)$, $u = fx$ and $v = fy$. We next show that A is a singleton. If not, then $\delta(A) > 0$, which implies $x \neq y$. Using $(*)$ we obtain

$$\delta(A) = d(fx, fy) < \delta\left(\bigcup_{h\in H_f} O(x,y,h)\right) \leq \delta(A),$$

a contradiction, and hence $A = \{w\}$ for some $w \in X$. Obviously, w is a fixed point of f. If z is another fixed point of f, then $z \in \bigcap_{n\in\omega} f^n X = A = \{w\}$; i.e., $z = w$. Hence w is the only fixed point of f.

Note that $f \in \bigcap_{g\in\mathcal{F}} C_g$. Thus we have $fgw = gfw = gw$ for $g \in \mathcal{F}$. Since f has a unique fixed point w, $gw = w$ for $g \in \mathcal{F}$; i.e., w is a fixed point of $\mathcal{F}$. This completes the proof.

In order to extend Janos' and Park's results to a mapping satisfying $(*)$, we need the following

Theorem (Meyers [7]). *Let f be a continuous self-mapping of a metric (X,d) with the following properties:*

(i) *f has a unique fixed point $w \in X$;*
(ii) *For any $x \in X$, $f^n x \to w$ as $n \to \infty$,*
(iii) *There exists an open neighborhood U of w with the property that given any open set V containing w there exists $k \in \mathbb{N}$ with $f^n U \subset V$ for $n > k$.*

Then for any $\alpha \in (0,1)$ there exists a metric $d' \in M(X)$ relative to which f satisfies $d'(fx, fy) \leq \alpha d'(x,y)$ for $x, y \in X$.

Theorem 2. *Let f be a continuos compact self-mapping of a bounded metric space (X,d) satisfying $(*)$. Then f has a unique fixed point, and furthermore, for any $\alpha \in (0,1)$ there exists a metric $d' \in M(X)$ relative to which f satisfies $d'(fx, fy) \leq \alpha d'(x,y)$ for all $x, y \in X$.*

Proof. Let $A = \bigcap_{n\in\omega} f^n X$. As in the proof of Theorem 1, we have $A = \{w\}$, which implies that (i) and (ii) of Meyers' theorem hold. To prove that (iii) holds we take $U = X$and observe that $f^{n+1}X \subset f^n X$, the diameter of which diminishes to zero as $n \to \infty$. Thus $f^n X$ squeezes into any neighborhood of w and the proof is complete.

The following simple example reveals that our Theorem 2 extends properly Theorem 1.1 of Janos [4] and Theorem 1 of Park [5].

Example. Let $X = \{1, 2, 4, 5, 8\}$ with the usual metric. Define a mapping $f : X \to X$ by $f1 = f4 = f5 = 5$, $f2 = 1$ and $f8 = 2$. Then f is a continuous compact self-mapping of X. It is easy to check that

$$d(fx,fy) \le 4 < 7 = \delta\left(\bigcup_{h\in H_f} O(x,y,h)\right)$$

for all $x, y \in X$ with $x \neq y$. Hence the conditions of Theorem 2 are satisfied. But Theorem 1.1 of Janos [4] and Theorem 1 of Park [5] are not applicable because

$$d(f2,f4) = 4 \not< 1 = \frac{1}{2}[d(2,f2)+d(4,f4)]$$

and

$$d(f2,f4) = 4 \not< 4 = \delta(O(2,4,f)).$$

Theorem 3. *Let $\mathcal{F}$ be a family of self-mappings of a complete metric space (X,d). Then the following statements are equivalent:*

(1) *$\mathcal{F}$ has a fixed point;*

(2) *There exists $x_0 \in X$ and a continuous condensing mapping $f \in \bigcap_{g\in\mathcal{F}} C_g$ such that $O(x_0,f)$ is bounded and*

$$d(fx,fy) < \delta(O(x,y,f)) \text{ for all } x,y \in X \text{ with } x \neq y;$$

(3) *There exists $x_0 \in X$ and $F \in \mathcal{F}$ and a continuous condensing mapping $f \in \bigcap_{g\in\mathcal{F}} C_g$ such that $O(x_0,f)$ is bounded and*

$$F(fx,fy) < \max\left\{F(x,y), F(x,fx), F(y,fy), \frac{F(x,fx)F(y,fy)}{F(x,y)}\right\}$$
$$\text{for all } x,y \in X \text{ with } x \neq y.$$

Proof. Let (1) hold and let w be a fixed point of $\mathcal{F}$. Define $f{:}X\to X$ by $fx = w$ for $x \in X$. Let $x_0 = w$ and $F \in \mathcal{F}$. It is easy to show that (2) and (3) hold.

Assume that (2) holds. Set $B = O(x_0,f)$ and $A = \bigcap_{n\in\omega} f^n\overline{B}$. Since f is condensing and $\alpha(B) = \max\{\alpha(\{x_0\}),\ \alpha(fB)\} = \alpha(fB)$, we have $\alpha(B) = 0$, which implies that B is precompact. Since X is complete, $\overline{B}$ is compact. By the continuity of f we get $f\overline{B} \subset \overline{fB} \subset \overline{B}$. As in the proof of Theorem 1, we conclude that A is a nonempty compact subset and $fA = A$. We assert that A is a singleton. Otherwise $\delta(A) > 0$. Since

A is compact and f maps A into itself, there exist $a, b, x, y \in A$ with $d(a,b) = \delta(A)$, $a = fx$, $b = fy$ and $x \neq y$. From (2) we have

$$\delta(A) = d(fx, fy) < \delta(O(x, y, f) \leq \delta(A),$$

a contradiction, and hence $A = \{w\}$ for some $w \in X$. Clearly, w is a fixed point of f. Suppose that f has another fixed point $v(\neq w)$, then by (2) we have

$$d(w, v) = d(fw, fv) < \delta(O(w, v, f)) = d(w, v)$$

which is impossible. Consequently w is a unique fixed point of f. It is easy to check that w is a fixed point of $\mathcal{F}$. Hence (1) holds.

Assume that (3) holds. Set $B = O(x_0, f)$. As above we may show that $\overline{B}$ is compact and f-invariant. Since the function F is lower semi-continuous, the function h defined by $hx = F(x, fx)$ for $x \in \overline{B}$ is lower semi-continuous and so assumes its minimum value at some $w \in \overline{B}$. Thus if $fw \neq w$, then $fw \in \overline{B}$ and

$$\begin{aligned} hfw &= F(fw, f^2w) \\ &< \max\left\{F(w, fw), F(w, fw), F(fw, f^2w), \frac{F(w, fw)F(fw, f^2w)}{F(w, fw)}\right\} \\ &= \max\{hw, hfw\} \end{aligned}$$

which implies that $hfw < hw$, contradicting the definition of w. It follows that w is a fixed point of f. If f has a second distinct fixed point u, then by (3) we have

$$\begin{aligned} F(w, u) &= F(fw, fu) \\ &< \max\left\{F(w, u), F(w, w), F(u, u), \frac{F(w, w)F(u, u)}{F(w, u)}\right\} = F(w, u), \end{aligned}$$

a contradiction. Therefore w is the only fixed point of f. It is a simple matter to show that w is a fixed point of $\mathcal{F}$. Hence (1) holds. This completes the proof.

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