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Nonflat pseudo-Riemannian space forms and homogeneous pseudo-Riemannian structures of class $\mathcal{S}_1$

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1. Introduction

The well-known characterization by AMBROSE and SINGER [1] of connected, simply connected and complete homogeneous Riemannian manifolds in terms of a (1,2) tensor field S on the manifold, which in turn generalizes Cartan's characterization of Riemannian symmetric spaces [3], has been extended in [5] to the pseudo-Riemannian case of any signature.

This extended characterization allows us to obtain (§2) a classification of homogeneous pseudo-Riemannian structures into eight classes in the pseudo-Riemannian case, according as the structure S belongs to an invariant subspace of certain space $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$, thus generalizing the Riemannian case studied in [7].

The main purpose of the present paper is to prove that if a connected, simply connected and complete pseudo-Riemannian manifold (M, g) is a nonflat pseudo-Riemannian space form, then (M, g) is locally isometric to a manifold which admits a nondegenerate homogeneous structure of class $\mathcal{S}_1$. The result follows by means of a Cayley transformation which we define here in terms of paracomplex numbers (for these numbers see [4,6]). The proof provides the pseudo-Riemannian models for any signature similar to the Riemannian Poincaré models.

We also prove, by using Cartan's moving frame method as in [8], that if a connected pseudo-Riemannian manifold (M, g) of any signature admits a nondegenerate homogeneous structure of class $\mathcal{S}_1$, then (M, g) is a nonflat pseudo-Riemannian space form.

2. Definitions and results

Let M be a connected C^∞ manifold of dimension $m+n$ endowed with a pseudo-Riemannian metric g of signature (m,n). Let ∇ denote the Levi-Civita connection of g and R the curvature tensor.

A homogeneous pseudo-Riemannian structure on (M,g) is [5] a tensor field S of type (1,2) on M such that the connection $\tilde{\nabla} = \nabla - S$ satisfies

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0 .$$

The following result is proved in [5]: *if (M,g) is connected, simply connected and complete, then it admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.* Notice that a homogeneous Riemannian manifold is always complete and reductive.

Let V be a finite dimensional real vector space endowed with an inner product $\langle\,,\rangle$ of signature (m,n), with the convention that (m,n) means m pluses and n minuses. $(V,\langle\,,\rangle)$ is the model for each tangent space T_xM, $x \in M$, of a reductive homogeneous pseudo-Riemannian manifold of signature (m,n). Consider the vector space $\mathcal{S}(V)$ of (0,3) tensors on $(V,\langle\,,\rangle)$ satisfying the same symmetries as a homogeneous pseudo-Riemannian structure S, that is,

$$\mathcal{S}(V) = \{S \in \overset{3}{\otimes} V^* : S_{XYZ} = -S_{XZY}\,,\ X,Y,Z \in V\}\,,$$

where $S_{XYZ} = \langle S_X Y, Z\rangle$. By using arguments similar to those in [7, §3] we can determine the decomposition of $\mathcal{S}(V)$ into subspaces which are invariant and irreducible under the action of the pseudo-orthogonal group $O(m,n)$ given by

$$(aS)_{XYZ} = S_{a^{-1}Xa^{-1}Ya^{-1}Z}\,, \quad a \in O(m,n)\,.$$

Specifically, being $c_{12}(S)(X) = \sum_i \varepsilon_i S_{e_i e_i X}$, where $\{e_i\}$ is an orthonormal basis of V, $\langle e_i, e_i\rangle = \varepsilon_i$, $\varepsilon_i = 1$ for $1 \le i \le m$, $\varepsilon_i = -1$ for $m+1 \le i \le m+n$, we have:

Theorem 2.1. *If $\dim V \geq 3$, then $\mathcal{S}(V)$ decomposes into the direct sum of subspaces which are invariant and irreducible under the action of $O(m,n)$:*

$$\mathcal{S}(V) = \mathcal{S}_1(V) \oplus \mathcal{S}_2(V) \oplus \mathcal{S}_3(V)\,,$$

where

$$\mathcal{S}_1(V) = \{S \in \mathcal{S}(V) : S_{XYZ} = \langle X,Y\rangle\omega(Z) - \langle X,Z\rangle\omega(Y),\ \omega \in V^*\}\,,$$

$$\mathcal{S}_2(V) = \{S \in \mathcal{S}(V) : \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0,\ c_{12}(S) = 0\}\,,$$

$$\mathcal{S}_3(V) = \{S \in \mathcal{S}(V) : S_{XYZ} + S_{YXZ} = 0\}\,.$$

If $\dim V = 2$ *then* $\mathcal{S}(V) = \mathcal{S}_1(V)$.

Proof. The representation theory of $O(m, n)$ is similar to that of $O(m+n)$ ([9], [2]), having only in mind that the trace maps are metric contractions and thus depend on the specific group $O(m, n)$. $\square$

Definition 2.1. A homogeneous pseudo-Riemannian structure S on (M, g) is said to be *of type* $\mathcal{S}_1$, if, at each point of $x \in M$, $S(x) \in \mathcal{S}(T_xM)$ belongs to $\mathcal{S}_1(T_xM)$. Let S be a homogeneous structure of type $\mathcal{S}_1$ on a connected pseudo-Riemannian manifold (M, g), that is, $S_{XYZ} = g(X, Y)\omega(Z) - g(X, Z)\omega(Y)$, where ω is a 1-form on M and let ξ be the dual vector field to ω, i.e. $g(X, \xi) = \omega(X)$. We say that S is *nondegenerate* if $g(\xi, \xi) \neq 0$.

Proposition 2.1. *Let (M, g) be a connected pseudo-Riemannian manifold which admits a nondegenerate homogeneous structure of type $\mathcal{S}_1$ defined by a vector field ξ. Then (M, g) is a nonflat pseudo-Riemannian space form with constant curvature $-g(\xi, \xi)$.*

Proof. Let S be a nondegenerate structure of type $\mathcal{S}_1$ and ξ the corresponding vector field. Let $\{e_i : i = 1, \dots, m+n\}$ be a local orthonormal frame on (M, g), where $g(e_i, e_j) = \varepsilon_j \delta_{ij}$, $\varepsilon_j = +1$ if $1 \leq j \leq m$, $\varepsilon_j = -1$ if $m+1 \leq j \leq m+n$, and let $\{\theta^i\}$ be the dual basis of $\{e_i\}$. If Ω^i_j and $\tilde{\Omega}^i_j$ are the curvature forms of ∇ and $\tilde{\nabla}$, respectively, where ∇ is the Levi-Civita connection of g and $\tilde{\nabla} = \nabla - S$, then, in a way similar to the one described in [8], we obtain $g(\xi, \xi)\tilde{\Omega}^i_j = 0$, where $g(\xi, \xi)$ is a nonzero constant, and $\Omega^i_j = \tilde{\Omega}^i_j + \varepsilon_j g(\xi, \xi)\theta^i \wedge \theta^j$. Since $g(\xi, \xi) \neq 0$ then $\Omega^i_j = \varepsilon_j g(\xi, \xi)\theta^i \wedge \theta^j$ and so (M, g) has constant curvature $-g(\xi, \xi)$. $\square$

Proposition 2.2. *Let (M, g) be a connected, simply connected and complete pseudo-Riemannian manifold of signature (m, n). If (M, g) is a pseudo-Riemannian model of nonzero constant curvature then it is locally isometric to a manifold which admits a nondegenerate homogeneous structure of type $\mathcal{S}_1$.*

Proof. If (M, g) has constant curvature $K \neq 0$ then (M, g) is locally isometric to the open subset

$$D = \left\{ (x_1, \dots, x_{m+n}) \in \mathbb{R}^{m+n} : 1 + \frac{K}{4} \sum_{i=1}^{m+n} \varepsilon_i x_i^2 > 0 \right\},$$

where $\varepsilon_i = +1$ if $1 \leq i \leq m$, and $\varepsilon_i = -1$ if $m+1 \leq i \leq m+n$, endowed with the pseudo-Riemannian metric

$$g_D = \frac{\sum_{i=1}^{m+n} \varepsilon_i dx_i^2}{\left(1 + \frac{K}{4} \sum_{i=1}^{m+n} \varepsilon_i x_i^2\right)^2}$$

(see [10, p. 69]). We shall construct a generalized Cayley transformation

$$c : D \to H^{m+n} = \{x \in \mathbb{R}^{m+n} : x_1 > 0\}$$

such that c is an isometry of (D, g_D) onto the half-space H^{m+n} endowed with the "Poincaré" metric

$$g_{H^{m+n}} = -\frac{1}{K} \frac{\sum_{i=1}^{m} du_i^2 - \sum_{i=1}^{n} dv_i^2}{u_1^2},$$

where we have denoted by (u_i, v_i) the coordinates in H^{m+n}.

For this, we can suppose $m < n$, by reversing if necessary the sign of the metric, and then embed $\mathbb{R}^{m+n}$ into $\mathbb{R}^{2n}$, and D into

$$D_{\mathbb{R}^{2n}} = \{x \in \mathbb{R}^{2n} : 1 + \frac{K}{4} \sum_{i=1}^{2n} \varepsilon_i x_i^2 > 0\},$$

where $\varepsilon_i = +1$ if $1 \leq i \leq n$, and $\varepsilon_i = -1$ if $n+1 \leq i \leq 2n$, endowed with the metric

$$g_{D_{\mathbb{R}^{2n}}} = \frac{\sum_{i=1}^{2n} \varepsilon_i dx_i^2}{\left(1 + \frac{K}{4} \sum_{i=1}^{2n} \varepsilon_i x_i^2\right)^2}.$$

We can now consider paracomplex coordinates $z_k = a_k + jb_k$ on $\mathbb{R}^{2n}$ (see Libermann [6], Cruceanu *et al.* [4]), and identifying $a_k = x_k$ for $k = 1, \ldots, n$; $b_k = x_k$ for $k = n+1, \ldots, 2n$, we can express the above metric as

$$g_{D_{\mathbb{R}^{2n}}} = \frac{\sum_{k=1}^{n} dx_k \cdot d\bar{z}_k}{\left(1 + \frac{K}{4} \sum_{k=1}^{n} z_k \bar{z}_k\right)^2}.$$

We recall that $j^2 = 1$, the conjugate element $\bar{z}_k$ of $z_k = a_k + jb_k$ is given by $\bar{z}_k = a_k - jb_k$, and $dz_k \cdot d\bar{z}_k = da_k^2 - db_k^2$.

Let $w_k = u_k + jv_k$ denote the paracomplex coordinates in $H^{2n} \subset \mathbb{R}^{2n}$ viewed as image of the Cayley transformation

$$\tilde{c} : D_{\mathbb{R}^{2n}} \to H^{2n} = \{w_k = u_k + jv_k \\ = (u_1, \dots, u_n, v_1, \dots, v_n) \in \mathbb{R}^{2n} : u_1 > 0\} ,$$

which we define by

$$w_1 = 2r \frac{(z_1 + 2r)(\bar{z}_1 - 2r) + \sum_{i=2}^{n} z_i \bar{z}_i}{(z_1 - 2r)(\bar{z}_1 - 2r) + \sum_{i=2}^{n} z_i \bar{z}_i}$$

and

$$w_k = \frac{8r^2 z_k}{(z_1 - 2r)(\bar{z}_1 - 2r) + \sum_{i=2}^{n} z_i \bar{z}_i} , \quad 2 \leq k \leq n ,$$

where $K = -1/r^2$.

Consider on H^{2n} the metric

$$g_{H^{2n}} = -\frac{1}{K} \frac{\sum_{i=1}^{n} dw_k \cdot d\bar{w}_k}{u_1^2} .$$

Then, as a long but straightforward computation shows, the transformation

$$\tilde{c} : (D_{\mathbb{R}^{2n}}, g_{D_{\mathbb{R}^{2n}}}) \to (H^{2n}, g_{H^{2n}})$$

is an isometry. From which it is immediate that the Cayley transformation

$$c : (D, g_D) \to (H^{m+n}, g_{H^{m+n}})$$

obtained by restricting $\tilde{c}$, is also an isometry.

Consequently, (M, g) is locally isometric to a pseudo-Riemannian "Poincaré" half-space, and then TRICERRI–VANHECKE's argument ([7, p. 55]) shows that $\xi = -Ku_1 \partial / \partial u_1$ is the vector field associated to a non-degenerate pseudo-Riemannian structure on H^{m+n}, denoted S, defined by

$$S_X Y = g(X, Y)\xi - g(Y, \xi)X, \quad X, Y \in \mathfrak{X}(H^{m+n}) ,$$

which holds $g(\xi, \xi) = -K \neq 0$. $\square$

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