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The "Two-Series Theorem" for symmetric random variables on nilpotent Lie groups

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Abstract. The classical "Three-Series Theorem" due to Kolmogorov is carried over to symmetric random variables on certain nilpotent Lie groups.

1. Introduction

The classical "Three-Series Theorem" for real valued random variables can be stated as follows: Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables and let $S_n = X_1+\cdots+X_n$. For $c>0$ let $X_{n,c} = X_n 1_{\{|X_n|<c\}}$ denote the truncated random variable. Then the almost sure (a.s.) convergence of $(S_n)_{n\geq 1}$ is equivalent to the convergence of the three series $\sum_{n\geq 1} P\{|X_n|\geq c\}$, $\sum_{n\geq 1} E(X_{n,c})$ and $\sum_{n\geq 1} V(X_{n,c})$, where E denotes the expectation and V the variance of a random variable. (See e.g. [6, Theorem IV.2.3].)

The aim of this note is to carry over this "Three-Series Theorem" to symmetric random variables with values in nilpotent Lie groups G: We show that for simply connected nilpotent Lie groups G the convergence of the series of tail probabilities and the truncated second moments of independent random variables X_n imply the almost sure convergence of the product $\prod_{n=1}^{\infty} X_n = X_1\cdot X_2\cdots$. If G is step 2-nilpotent it turns out that these conditions are also necessary for the almost sure convergence of the product.

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Examples of simply connected (step 2-) nilpotent Lie groups are the *Heisenberg groups* $\mathbb{H}$ given as $\mathbb{R}^{2d+1} \cong \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ with the product

$$x \cdot y = (x' + y', x'' + y'', x''' + y''' + \frac{1}{2}(\langle x', y''\rangle - \langle x'', y'\rangle)) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$$

for $x = (x', x'', x''')$, $y = (y', y'', y''') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. The so-called *groups of type H* (cf. [4]) are all simply connected step 2-nilpotent.

Using the Campbell–Hausdorff formula, a simply connected nilpotent Lie group G can be realized as $G = \mathbb{R}^d$ for some non-negative integer d equipped with the multiplication

$$(*) \quad x \cdot y = P(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([[x, y], y] + [[y, x], x]) + \cdots,$$

where $P : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a polynomial mapping in the components of x and y. (See [1, (1.2) Proposition].) Clearly, the neutral element e of G is 0 and $x^{-1} = -x$ for every $x \in G$. G is said to be nilpotent of step $r \geq 0$, if for the lower central series of G: $G_{(1)} \stackrel{\text{def}}{=} G$, $G_{(j)} \stackrel{\text{def}}{=} [G, G_{(j-1)}]$ we have $G_{(r+1)} = \{0\}$. Then it follows from the Campbell–Hausdorff formula that the polynomial mapping P in $(*)$ is of degree $\leq r$.

Now let $(X_n)_{n\geq 1}$ be a sequence of independent G-valued random variables with probability distributions $(\nu_n)_{n\geq 1}$ and let $S_N \stackrel{\text{def}}{=} \prod_{n=1}^{N} X_n = X_1 \cdot X_2 \cdots X_N$ denote the partial product. Since by [3, XII.3 Theorem 2.3] every simply connected nilpotent Lie group is aperiodic (cf. [2, 2.2.18 Definition]), the convergence of $(S_N)_{N\geq 1}$ in probability, almost surely, and in distribution (i.e. the weak convergence of $(\nu_1 * \cdots * \nu_N)_{N\geq 1}$) resp. are equivalent. (See [2, 2.2.19 Theorem].) Using this fact, for the sufficiency part of our theorem it is enough to show that our conditions imply the convergence in distribution of $(S_N)_{N\geq 1}$. We will do this by using Fourier analytic methods on G, especially Lévy's continuity theorem.

2. Results

Let G be a simply connected nilpotent Lie group. A G-valued random variable X is called symmetric, if X and $X^{-1} = -X$ have the same distribution. Let $\mathcal{L}(X)$ denote the law of a random variable X. For $c > 0$ let $X_c \stackrel{\text{def}}{=} X 1_{\{\|X\| < c\}}$ denote the truncated random variable, where $\|X\|$ is the Euclidean norm of X. (Note that we have realized G

as $\mathbb{R}^d$.) Furthermore for $x_1, x_2, \ldots \in G$ we define $\prod_{n=1}^{N} x_n = x_1 \cdot x_2 \ldots x_N$ and $\prod_{n=1}^{\infty} x_n = \lim_{N\to\infty} \prod_{n=1}^{N} x_n$. We will prove

Theorem 1. *Let G be a simply connected nilpotent Lie group and let $(X_n)_{n\geq 1}$ be a sequence of independent symmetric G-valued random variables.*

(a) *If for some $c > 0$*

$$\sum_{n=1}^{\infty} P\{\|X_n\| \geq c\} < \infty \tag{1}$$

and

$$\sum_{n=1}^{\infty} E\|X_{n,c}\|^2 < \infty\,, \tag{2}$$

then $\prod_{n=1}^{\infty} X_n$ is a.s. convergent.

(b) *If furthermore G is step 2-nilpotent and $\prod_{n=1}^{\infty} X_n$ is a.s. convergent, then (1) and (2) hold for every $c > 0$.*

Remark. It is easy to see that if condition (1) and (2) hold for some $c > 0$, then they hold for any $c > 0$. Hence we may assume that c is small enough.

First we need an auxiliary result.

Lemma 1. *Under the conditions (1) and (2) of the Theorem the sequence $\Big(\mathcal{L}\big(\prod_{n=1}^{N} X_n\big)\Big)_{N\geq 1}$ is weakly relatively compact.*

Proof. Assume $c \in]0,1]$ small enough. We show that the sequence $\Big(\prod_{n=1}^{N} X_{n,c}\Big)_{N\geq 1}$ is L^2-bounded (with respect to $\|\cdot\|$); then $\Big(\mathcal{L}\big(\prod_{n=1}^{N} X_{n,c}\big)\Big)_{N\geq 1}$ is weakly relatively compact and the assertion follows from condition (1) and the Borel–Cantelli Lemma. Write every component of $[x,y]$ in the form $x^{\mathrm{tr}}\cdot A\cdot y$ with a suitable matrix A. Consider the expansion $E(Q)$ of $E||\prod_{n=1}^{N} X_{n,c}||^2$ as expectation of a polynomial in the elements of the matrices A and in the components of the $X_{n,c}$. Now, for Q perform the following procedure (P) (in the prescribed order):

i) Delete any monomial where there is an n such that only one component of $X_{n,c}$ occurs, and actually in first power.

ii) Replace any element of one of the matrices A by its absolute value and any component on an $X_{n,c}$ by $||X_{n,c}||$.

iii) Replace any exponent (> 0) of a power of an $||X_{n,c}||$ by 2.
By the symmetry, we have $EX_{n,c} = 0$, so i) does not change the value of $E(Q)$. Clearly, ii) does not decrease the value of Q. By i), no exponent 1 remains before iii), so (since $c \leq 1$) also iii) does not decrease the value of Q. Hence the procedure (P) yields an upper bound $E(Q')$ for $E(Q)$. Now the degree of Q' in the $||X_{n,c}||$ is bounded (as $N \to \infty$) and so it is (by the nilpotency) in the absolute values of the elements of the matrices A. So (by the independence) $E(Q')$ may be majorized by a constant times a fixed power of $\sum_{n=1}^{\infty} E||X_{n,c}||^2$, hence by condition (2) $E(Q')$ (and thus $E(Q)$) is bounded (as $N \to \infty$), which proves the asserted L^2-boundedness.

To fix notation we now recall some basic facts about Fourier analysis on Lie groups (see [5, p. 115–118] for details):

Let $\boldsymbol{g} = \mathbb{R}^d$ denote the Lie algebra of G with basis $\{Y_i = e_i : i = 1, \dots, d\}$, where $\{e_i\}_{i=1\dots d}$ is the natural basis of $\mathbb{R}^d$. As usual we regard every element $Y \in \boldsymbol{g}$ as a left invariant differential operator on G. Furthermore let $\mathrm{Irr}(G)$ denote the set of all irreducible unitary representations of G. For $D \in \mathrm{Irr}(G)$ let $\mathcal{H}(D)$ be the representation Hilbert space of D with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A vector $u \in \mathcal{H}(D)$ is said to be a C^∞-vector (of D) if the function $x \mapsto \langle D(x)u, v \rangle$ is C^∞ for all $v \in \mathcal{H}(D)$. We denote the subspace of all C^∞-vectors by $\mathcal{H}_0(D)$. It is well known that $\mathcal{H}_0(D)$ is a dense subspace of $\mathcal{H}(D)$. For a probability measure ν on G we define its Fourier transform $\hat{\nu}$ by

$$\langle \hat{\nu}(D)u, v \rangle \stackrel{\text{def}}{=} \int_G \langle D(x)u, v \rangle \, d\nu(x)$$

for all $D \in \mathrm{Irr}(G)$ and all $u, v \in \mathcal{H}(D)$. In this context some of the usual properties of characteristic functions, especially a continuity theorem hold.

Proof of Theorem 1. First we show that conditions (1) and (2) imply

$$\sum_{n=1}^{\infty} \|\hat{\nu}_n(D)u - u\| < \infty \tag{3}$$

for all $D \in \mathrm{Irr}(G)$ and all $u \in \mathcal{H}_0(D)$, where ν_n is the distribution of X_n. In fact consider the symmetric open neighborhood $U_c \stackrel{\text{def}}{=} \{x \in G : \|x\| < c\}$

of $0 \in G$. Following [5, Lemma 5.1] we have for $D \in \mathrm{Irr}(G)$ and $u \in \mathcal{H}_0(D)$:

$$D(x)u - u = \sum_{i=1}^{d} x_i D(Y_i)u + \frac{1}{2} \sum_{i,j=1}^{d} x_i x_j T(D)(x) D(Y_i) D(Y_j) u$$

for all $x \in U_0$. Here each $T(D)(x)$ is a linear contraction (i.e. a bounded linear operator on $\mathcal{H}(D)$ such that $\|T(D)(x)\| \le 1$). For $u \in \mathcal{H}_0(D)$ let $\|u\|_* = \|u\| + \sum_{i,j=1}^{d} \|D(Y_i)D(Y_j)u\|$. Using $|x_i x_j| \le \sum_{i=1}^{d} x_i^2 = \|x\|^2$ for all $x \in U_c$ and the symmetry of the X_n we get

$$\begin{aligned}\Big\| \int_{U_c} (D(x)u - u)\, d\nu_n(x) \Big\| &\le \frac{1}{2} \sum_{i,j=1}^{d} \int_{U_c} \|x\|^2 \, d\nu_n(x) \|D(Y_i)D(Y_j)u\| \\ &\le C\|u\|_* E\|X_{n,c}\|^2 ,\end{aligned}$$

for some constant $C > 0$. On the other hand

$$\Big\| \int_{\complement U_c} (D(x)u - u)\, d\nu_n(x) \Big\| \le 2\|u\| P\{\|X_n\| \ge c\} \le 2\|u\|_* P\{\|X_n\| \ge c\} ,$$

so we finally conclude

$$\begin{aligned}\|\hat{\nu}_n(D)u - u\| &\le \Big\| \int_{U_c} (D(x)u - u)\, d\nu_n(x) \Big\| + \Big\| \int_{\complement U_c} (D(x)u - u)\, d\nu_n(x) \Big\| \\ &\le C\|u\|_* (E\|X_{n,c}\|^2 + P\{\|X_n\| \ge c\})\end{aligned}$$

and (3) follows from (1) and (2).

Since by Lemma 1 $(\mathcal{L}(S_N))_{N \ge 1}$ is weakly relatively compact we have, using the continuity theorem of the Fourier transform (see [5, p. 117 and Lemma 2.1]), only to show that

$$\hat{\mu}_N(D)u = \prod_{n=1}^{N} \hat{\nu}_n(D)u$$

is convergent in $\mathcal{H}(D)$ for every $u \in \mathcal{H}_0(D)$ and every $D \in \mathrm{Irr}(G)$. But $\|\hat{\nu}_n(D)\| \le 1$ and so

$$\begin{aligned}\Big\| \prod_{n=1}^{N} \hat{\nu}_n(D)u - \prod_{n=1}^{N+K} \hat{\nu}_n(D)u \Big\| &= \Big\| \prod_{n=1}^{N} \hat{\nu}_n(D)\big(I - \prod_{n=N+1}^{N+K} \hat{\nu}_n(D)\big)u \Big\| \\ \le \Big\| u - \prod_{n=N+1}^{N+K} \hat{\nu}_n(D)u \Big\|\end{aligned}$$

$$
\begin{aligned}
&= \big\| \big((\hat\nu_{N+1}(D) - I) + \hat\nu_{N+1}(D)(\hat\nu_{N+2}(D) - I) + \dots \\
&\qquad \dots + \hat\nu_{N+1}(D) \cdots \hat\nu_{N+K-1}(D)(\hat\nu_{N+K}(D) - I) \big) u \big\| \\
&\le \sum_{n=N+1}^{N+K} \|\hat\nu_n(D)u - u\| \,.
\end{aligned}
$$

Using the above estimation and (3) we conclude that $(\hat\mu_N(D)u)_{N\ge1}$ is a Cauchy sequence and hence convergent in $\mathcal{H}(D)$. This completes the proof of the first part of our Theorem.

For the proof of the second part we assume that G is step 2-nilpotent and $\prod_{n=1}^{N} X_n$ is a.s. convergent. By the symmetry, the processes

$$
\left\{ \prod_{n=1}^{N} X_n \right\}_{N\ge1} \qquad , \qquad \left\{ -\prod_{n=1}^{N} X_{N+1-n} \right\}_{N\ge1}
$$

have the same distribution, (since $-\prod_{n=1}^{N} X_{N+1-n} = \prod_{n=1}^{N} -(X_n)$), so $\{-\prod_{n=1}^{N} X_{N+1-n}\}_{N\ge1}$ is a.s. a Cauchy sequence, hence

$$
\lim_{N\to\infty} \prod_{n=1}^{N} X_{N+1-n}
$$

and thus

$$
\lim_{N\to\infty} \left(\prod_{n=1}^{N} X_n + \prod_{n=1}^{N} X_{N+1-n} \right)
$$

exist a.s. But

$$
\begin{aligned}
\prod_{n=1}^{N} X_n + \prod_{n=1}^{N} X_{N+1-n} &= \sum_{n=1}^{N} X_n + \frac12 \sum_{1\le n<m\le N} [X_n, X_m] + \sum_{n=1}^{N} X_{N+1-n} \\
&\quad + \frac12 \sum_{1\le n<m\le N} [X_{N+1-n}, X_{N+1-m}] \\
&= 2\sum_{n=1}^{N} X_n + \frac12 \sum_{1\le n<m\le N} ([X_n, X_m] + [X_m, X_n]) \\
&= 2\sum_{n=1}^{N} X_n \,,
\end{aligned}
$$

hence condition (1) and (2) hold for every $c > 0$ by the classical Three-Series Theorem in the vector space case.

Remark. Interestingly enough, our Theorem implies that if G is step 2-nilpotent and $\prod_{n=1}^{\infty} X_n$ is convergent, then an arbitrary reordering of the X_n does not disturb the convergence, i.e. $\prod_{n=1}^{\infty} X_{\sigma(n)}$ is convergent for every permutation σ of $\mathbb{N}$. It would be interesting to investigate, for a specific sequence $(X_n)_{n\geq 1}$, what the class of possible limit laws of $\prod_{n=1}^{\infty} X_{\sigma(n)}$ (σ any permutation of $\mathbb{N}$) looks like.

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