On the distribution of real roots of almost-periodical polynomials.

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1. It was one of the first questions of algebra, to draw conclusions about the number of real zeros in an interval of a rational polynomial with real coefficients; I mention only the rules of DESCARTES—HARRIOT, JACOBI, BUDAN—FOURIER and STURM. Concerning the new literature we mention only the papers of LAGUERRE, PÓLYA, FEKETE, I. SCHUR, SZEGŐ, SCHOENBERG, OBRESCHKOFF, LITTLEWOOD—OFFORD and LIPKA. It was LAGUERRE[1]), who investigated the case when the exponents of the polynomial are not necessarily integers; he proved e. g. the analogon of DESCARTES—HARRIOT's rule for polynomials

$$f(x)=\sum_{\nu=0}^{n} a_\nu x^{\lambda_\nu},\ 0=\lambda_0<\lambda_1<\ldots<\lambda_n$$

or — and this is the same — for the functions

$$g(x)=\sum_{\nu=0}^{n} a_\nu e^{\lambda_\nu x} \tag{1, 1}$$

It is an interesting question how to generalise theorems concerning rational polynomials to polynomials of the form (1, 1); in this connection I mention only the recent investigations of L. SCHWARTZ[2]).

2. A parallel theory has been developed relating to the cosine-polynomial

$$h(x)=\sum_{\nu=0}^{n} a_\nu \cos \nu x$$

resp to the general trigonometrical polynomial

$$l(x)=\sum_{\nu=0}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

Some investigations in the theory of zeta-function of RIEMANN led to the question, to find an upper bound as good as possible of the number $N(H, a, d)$ of the real zeros of the almost periodic cosine-polynomial

$$H(x)=\sum_{\nu=0}^{n} a_\nu \cos \lambda_\nu x \tag{2, 1}$$

$$0=\lambda_0<\lambda_1<\ldots<\lambda_n, \qquad a_\nu \text{ real}$$

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1) E. LAGUERRE: Ouvres. Bd. I. p. 3.

2) L. SCHWARTZ: "Etude de sommes exponentielles reélles". *Actualités Scientifiques et Industrielles* (1943).

in the interval

$$a-d \leqq x \leqq a. \tag{2, 2}$$

In the classical case when λ_ν's are integers, we have obviously, if $h(x) \not\equiv 0$,

$$N(h, a, d) \leqq \left(1+\left[\frac{d}{\pi}\right]\right)\lambda_n.$$

In the case, when

$$a-d=-T,\ a=T, \qquad T \text{ large}$$

$N(H, T, 2T)$ has been investigated by M. KAC[3]) who proved in the case of linearly independent λ's that

$$\lim_{T\to\infty} \frac{N(H, T, 2T)}{2T}$$

exists and can be expressed as a double infinite integral, whose integrand contains infinite products. This expression depends upon all coefficients and exponents in a complicated way. It is somewhat surprising that — anyway in the case of positive coefficients — an estimation of the type

$$N(H, a, d) \leqq C_0(a, d, n, \lambda_n)$$

holds *independently* of $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ (the λ's are not necessarily linearly independent any more) and the numerical values of the coefficients. More exactly we shall prove the following

Theorem I. *For the number $N(H, a, d)$ of the real zeros of the polynomial*

$$H(x)=\sum_{\nu=0}^{n} a_\nu \cos \lambda_\nu x, \quad 0=\lambda_0<\lambda_1<\ldots<\lambda_n$$

in $a-d \leqq x \leqq a$ we have — if all the coefficients are positive — the estimation

$$N(H, a, d) \leqq 6n \log \frac{24(|a|+d)}{d} + 6d\lambda_n. \tag{2, 3}$$

3. It is natural to raise the analogous question for the polynomial

$$R(x)=\sum_{\nu=1}^{n} b_\nu \sin \lambda_\nu x.$$

For this polynomial we have

Theorem II. *If the coefficients of $R(x)$ are positive, we have*

$$N(R, a, d) \leqq 1 + 6n \log \frac{24(|a|+d)}{d} + 6d\lambda_n. \tag{3, 1}$$

As the examples $H(x)=\cos \lambda x$ and $R(x)=\sin \lambda x$ show, the estimations (2, 3) and (2, 4) are in a sense not far from being best-possible. The very interesting question as to what can be said removing the restriction the coefficients being positive remains unsettled as well as the analogous questions for the polynomial

[3]) M. KAC: "On the distribution of values of trigonometric sums with linearly independent frequencies", *Amer. J. Math.*, **65** (1943), p. 609–615.

$$L(x)=\sum_{\nu=0}^{n}(a_\nu\cos\lambda_\nu x+b_\nu\sin\lambda_\nu x).$$

If for $L(x)$ there is an estimation of type (2, 3) independently of the sign of its coefficients then replacing x by $x+a$ we obtain even

$$N(L,a,d)\leqq C_1(d,n,\lambda_n)$$

what is really the best one can expect.

As to the complex zeros of $L(x)$ see[4]).

What we can actually prove, is the estimation

$$N(H,a,d)\leqq 6n\left\{\log\frac{24(|a|+d)}{d}+\log\frac{|a_0|+\ldots+|a_n|}{|a_0+\ldots+a_n|}\right\}+6d\lambda_n,$$

without any restriction on the coefficients. Hence if e. g.

$$\frac{|a_0|+\ldots+|a_n|}{|a_0+\ldots+a_n|}<e^n,$$

we obtain the inequality (2, 3) again essentially.

4. First we reduce the proof of theorem I to the case, when $a\geqq d$. Suppose we proved for $a\geqq d$

$$N(H,a,d)\leqq(2n+1)\log\frac{24a}{d}+3d\lambda_n. \tag{4, 1}$$

If $a<d$, $a>0$ then

$$N(H,a,d)\leqq N(H,a+d,3d)$$

and applying (2, 3) to the intervals $0\leqq x\leqq 2d-a$ resp. $0\leqq x\leqq a+d$ we obtain in this case

$$N(H,a,d)\leqq 2N(H,2d,2d)\leqq 2(2n+1)\log 24+6d\lambda_n. \tag{4, 2}$$

If $a\leqq 0$ then, since $H(x)$ is even, we have

$$N(H,a,d)=N(H,|a|+d,d)$$

i. e.

$$N(H,a,d)\leqq(2n+1)\log\frac{24(|a|+d)}{d}+3d\lambda_n. \tag{4, 3}$$

From (4, 1), (4, 2) and (4, 3) we have finally

$$N(H,a,d)\leqq 2(2n+1)\log\frac{24(|a|+d)}{d}+6d\lambda_n\leqq 6n\log\frac{24(|a|+d)}{d}+6d\lambda_n,$$

i. e. theorem I. will be proved, if (4, 1) will be.

5. The proof of the inequality (4, 1) is based upon a lemma which I used for various purposes[5]). This runs as follows.

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4) G. Pólya: Untersuchungen über Lücken und Singularitäten von Potenzreihen. *Math. Zeitschrift* Bd. 29 (1929), p. 549–640, esp. p. 594. Footnote.

5) P. Turán: "On the gap-theorem of Fabry". *Acta Hung. Math.* (1947), p. 21–29. — "Sur la theorie des fonctions quasi analytiques". *Comptes Rendus* 1947, p. 1750–1752. — "On Riemanns hypothesis". *Bull. de l'Acad. des Sciences de l'URSS.* p. 197–262. In the first two and last of these papers one can find a simple proof of this lemma. As P. Erdős pointed out to me one can replace this lemma by the classical approximation theorem of Dirichlet; of course the upper bound becomes then much worse.

If $|z_1| \geqq 1, |z_2| \geqq 1, \ldots, |z_N| \geqq 1$ and $m \geqq N$ then

$$\max_{m-N \leqq y \leqq m} |c_1 z_1^y + \ldots + c_N z_N^y| > \left(\frac{N}{24m}\right)^N |c_1 + \ldots + c_N|. \tag{5, 1}$$

Replacing N by $(2n+1)$, m by $\frac{a}{d}(2n+1)$, y by $x\frac{2n+1}{d}$, further

$$z_1 = 1,\ z_2 = e^{i\frac{d}{2n+1}\lambda_1},\ z_3 = e^{-i\frac{d}{2n+1}\lambda_1}, \ldots, z_{2n+1} = e^{-i\frac{d}{2n+1}\lambda_n}$$

$$c_1 = a_0,\ c_2 = \frac{a_1}{2},\ c_3 = \frac{a_1}{2}, \ldots, c_{2n+1} = \frac{a_n}{2}$$

we obtain

$$\max_{a-d \leqq x \leqq a} \left|\sum_{\nu=0}^{n} a_\nu \cos \lambda_\nu x\right| > \left(\frac{d}{24a}\right)^{2n+1} |a_0 + \ldots + a_n|. \tag{5, 2}$$

Let $x = x_1$ be the value for which this absolute maximum of $H(x)$ with respect to $a-d \leqq x \leqq a$ is assumed The quantity $N(H, a, d)$ is obviously not larger than the number K of the zeros of $H(z)$ in the circle $|z - x_1| \leqq d$. Hence estimating K by JACOBI--JENSEN's inequality we obtain

$$N(H, a, d) \leqq \log \max_{|z-x_1| \leqq ed} \frac{|H(z)|}{|H(x_1)|}$$

and applying (5, 2)

$$N(H, a, d) \leqq (2n+1) \log \frac{24a}{d} + \log \max_{|z-x_1| \leqq ed} \frac{|H(z)|}{|a_0 + \ldots + a_n|}.$$

Since -- owing to the positivity of the a_ν's —

$$|H(z)| \leqq (|a_0| + |a_1| + \ldots + |a_n|)\, e^{\lambda_n e d} = |a_0 + \ldots + a_n|\, e^{\lambda_n e d},$$

we have

$$N(H, a, d) < (2n+1) \log \frac{24a}{d} + 3d\lambda_n. \qquad \text{(Q. e. d.)}$$

6. The proof of theorem II cannot be done similarly to that of theorem I. But we can reduce to it remarking that applying ROLLE's theorem we have

$$N(R, a, d) \leqq 1 + N\left(\frac{dR}{dx}, a, d\right).$$

Since $\frac{dR}{dx}$ is a polynomial of type (2, 1) and its coefficients are again positive, we can apply theorem I and we obtain

$$N(R, a, d) \leqq 1 + 6n \log \frac{24(|a| + d)}{d} + 6d\lambda_n.$$

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