On the smallest convex cover of a simple arc of space-curve.

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A bounded convex point set may be defined as a point set which is identical to the set of its chords. More explicitely, if $\omega(S)$ denotes the set of all points contained at least in one chord (= segment of straight line joining two points of S) of the point set S, then S is convex if and only if

(1) $$\omega(S)=S.$$

Adopting this definition of the process ω, the smallest convex cover $\Omega(S)$ of a bounded point set S in the *three*-dimensional space R_3 is evidently given by

(2) $$\Omega(S)=\omega(\omega(S)),$$

while in general $\Omega(S)\neq\omega(S)$.

Considering that in general the iteration of the process ω is necessary, it is natural to ask, whether there are point sets whose smallest convex cover coincides with the set of their chords, i. e. for which

(3) $$\Omega(S)=\omega(S)$$

holds.

The question as to the validity of (3) is easily decided in the case when the set $\omega(S)$ of the chords of S is convex. Indeed in this case we infer from (1) and (2) that

$$\Omega(S)=\omega(\omega(S))=\omega(S).$$

A well-known example of a set S, for which $\Omega(S)=\omega(S)$ holds is a pair of convex point sets in R_3. In this example however every interior point of $\Omega(S)$ is contained in an infinity of chords of S.

In the present note I shall discusse a classe $\{\gamma\}$ of space-curves in R_3 which possess the following two properties

I $\Omega(\gamma)=\omega(\gamma)$,

II *every interior point of* $\Omega(\gamma)$ *is contained in one and only one chord of* γ.

It is obvious that a space-curve containing four coplanar points cannot possess the property II All the more surprising seems to be the fact, (see p. 69) that if γ contains no four coplanar points then it possesses both properties I and II. Thus one of the results of the present paper is that the properties I and II together is equivalent to the fact that γ does not contain four coplanar points. The arcs of space-curves not containing four coplanar points, or according to the terminology of HJELMSJEV, the *simple* arcs of space-curves may be regarded as the three-dimensional analoga of the convex plane-curves (not containing three collinear points). Their properties stated in I and II admit the following more intuitive formulation:

The smallest convex cover of a simple arc of space-curve is filled up simply and without gap by its chords.

This result enables us to establish a simple formula for the calculation of an integral extended over the smallest convex cover of a simple arc of space-curve. Let us consider a simple arc given by the equations

$$x=x(t),\ y=y(t),\ z=z(t);\ a\leqq t\leqq b,$$

where $x(t)$, $y(t)$, $z(t)$ have continuous first derivatives with respect to t. Let us coincide the axis z with the terminating chord AB of the arc (i. e. the chord joining the terminating points of tne arc):

$$x(a)=x(b)=y(a)=y(b)=0;\ z(a)<z(b). \tag{4}$$

Any point of the chord joining $[x(t_1), y(t_1), z(t_1)]$ and $[x(t_2), y(t_2), z(t_2)]$ may be specified by the equations

$$\begin{aligned} x&=\frac{1+t_3}{2}x(t_1)+\frac{1-t_3}{2}x(t_2\\ y&=\frac{1+t_3}{2}y(t_1)+\frac{1-t_3}{2}y(t_2)\\ z&=\frac{1+t_3}{2}z(t_1)+\frac{1-t_3}{2}z(t_2), \end{aligned} \tag{5}$$

and in consequence of the property II these equations establish a $1-1$ correspondence between the interior points of $\Omega(\gamma)$ and the interior points of the region characterised by the inequalities

$$a\leqq t_1<t_2\leqq b;\ -1\leqq t_3\leqq 1. \tag{6}$$

Hence, owing to (6)

$$\iiint\limits_{\Omega(\gamma)} f(x,y,z)\,dx\,dy\,dz=\frac{1}{2}\int\limits_a^b\int\limits_a^b\int\limits_{-1}^{+1}\bar{f}(t_1,t_2,t_3)\,\frac{\partial(x,y,z)}{\partial(t_1,t_2,t_3)}\,dt_1\,dt_2\,dt_3.$$

Especially for the volume V of the smallest convex cover of γ we get

$$V=\iiint\limits_{\Omega(\gamma)} dx\,dy\,dz=\frac{1}{2}\int\limits_{-1}^{+1}\int\limits_a^b\int\limits_a^b\frac{\partial(x,y,z)}{\partial(t_1,t_2,t_3)}\quad t_1\,dt_1\,dt_3=$$

$$=\frac{1}{16}\int\limits_a^b\int\limits_a^b\begin{vmatrix}\dot{x}(t_1), & \dot{x}(t_2), & x(t_1)-x(t_2)\\ \dot{y}(t_1), & \dot{y}(t_2), & y(t_1)-y(t_2)\\ \dot{z}(t_1), & \dot{z}(t_2), & z(t_1)-z(t_2)\end{vmatrix}dt_1\,dt_2.\int\limits_{-1}^{+1}(1+t_3)(1-t_3)\,dt_3=$$

$$=\frac{1}{12}\int_a^b\int_a^b\begin{vmatrix}\dot{x}(t_1), & \dot{x}(t_2), & x(t_1)\\ \dot{y}(t_1), & \dot{y}(t_2), & y(t_1)\\ \dot{z}(t_1), & \dot{z}(t_2), & z(t_1)\end{vmatrix}dt_1\,dt_2-\frac{1}{12}\int_a^b\int_a^b\begin{vmatrix}\dot{x}(t_1), & \dot{x}(t_2), & x(t_2)\\ \dot{y}(t_1), & \dot{y}(t_2), & y(t_2)\\ \dot{z}(t_1), & \dot{z}(t_2), & z(t_2)\end{vmatrix}dt_1\,dt_2.$$

Furthermore changing t_1 and t_2 we obtain according to (4)

(7) $$V=\frac{1}{6}\int_a^b\int_a^b\begin{vmatrix}\dot{x}(t_1), & \dot{x}(t_2), & x(t_1)\\ \dot{y}(t_1), & \dot{y}(t_2), & y(t_1)\\ \dot{z}(t_1), & \dot{z}(t_2), & z(t_1)\end{vmatrix}dt_1\,dt_2=$$

$$=-\frac{1}{6}\int_a^b\dot{x}(t_2)\,dt_2.\int_a^b\begin{vmatrix}\dot{y}(t_1), y(t_1)\\ \dot{z}(t_1), z(t_1)\end{vmatrix}dt_1+\frac{1}{6}\int_a^b\dot{y}(t_2)\,dt_2.\int_a^b\begin{vmatrix}\dot{x}(t_1), x(t_1)\\ \dot{z}(t_1), z(t_1)\end{vmatrix}dt_1-$$

$$-\frac{1}{6}\int_a^b\dot{z}(t_2)\,dt_2.\int_a^b\begin{vmatrix}\dot{x}(t_1), x(t_1)\\ \dot{y}(t_1), y(t_1)\end{vmatrix}dt_1=\frac{z(b)-z(a)}{6}\int_a^b\begin{vmatrix}x(t) & y(t)\\ \dot{x}(t) & \dot{y}(t)\end{vmatrix}dt.$$

As an application I shall discusse a special case of the following maximum-problem referring to space-curves which is as far as I know not yet solved hitherto. What is the maximum of the volume of the smallest convex covers of all arcs of space-curves which have a prescribed length L?

If we restrict ourselves to the subset of all *simple* arcs of space-curves having the length L, then the maximum of the volume will be easily found to be

(8) $$V_{\max}=\frac{L^3}{18\sqrt{3}\pi},$$

and this maximum is reached only for the arc of helix given by the equations

$$x=\frac{L}{\sqrt{6}\pi}\cos t$$
$$y=\frac{L}{\sqrt{6}\pi}\sin t \qquad 0\leqq t\leqq 2\pi.$$
$$z=\frac{L}{2\sqrt{3}\pi}t$$

* * *

Let be γ a simple arc with the terminating chord AB, suppose that γ has at each of its points P a tangent which varies continuously with P, and consider the cone which projects γ from a point C collinear with A and B and separated form A by B.

This projecting cone is closed (i. e. its line of intersection with a sphere having its centre at the vertex is a closed curve) because the projecting lines of the terminating points A and B coincide.

Apart from the double generator-line passing through A and B this cone cannot contain any more double (or multiple) generator-line. Indeed, a double generator-line would contain two points of γ, which were coplanar with A and B, contrary to our assunption on γ.

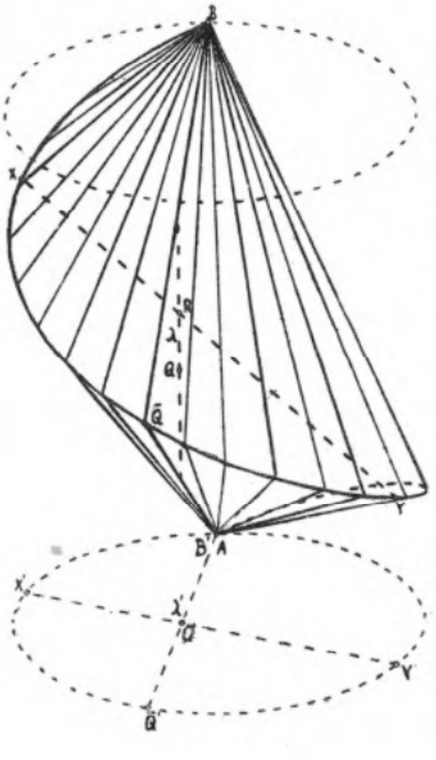
Further the closed projecting cone must be convex (i. e. its line of intersection with any plane must be a convex curve), because in the contrary case the cone would have four coplanar generators and the points of γ projected by these four lines would be coplanar, contrary to our assumption.

Obviously the same statement holds if the point C, being collinear with A and B, is separated from B by A.

In this way it is proved that from any point of the extension of the terminating chord the simple arc γ is projected by a closed convex cone having only one double generator-line.

From this result we infer that the projecting cylinder of γ parallel to AB is a closed convex cylinder.

The cones which project the arc γ from one of its terminating points A, B, are not determined and their rôle will be assumed by two cones which will turn out to be the supporting cones (Stützkegel) of γ at A and B. The cone at A consists of all lines projecting the interior points of γ from A, and of an angular region of the supporting plane passing through AB and the tangent of γ at A. This cone and the corresponding one at B are both closed, convex cones in consequence of the same considerations as discussed above, and thus they constitute indeed the supporting cones of γ at A and B.



Let now the convex point set consisting of all points interior to both supporting cones be denoted by $D(\gamma)$. I assert that $D(\gamma)$ is identical to the smallest convex cover of γ.

To prove this assertion I will first show that

$$\omega(\gamma) = D(\gamma),$$

further, $D(\gamma)$ being convex

$$\omega(D(\gamma)) = D(\gamma)$$

and hence we draw the conclusion that

$$\Omega(\gamma) = \omega(\omega(\gamma)) = \omega(D(\gamma)) = D(\gamma).$$

In order to prove the relation $\omega(\gamma) = D(\gamma)$ take an arbitrary point Q in the interior of $D(\gamma)$ Draw the plane QAB, which cuts γ in a single point $\bar{Q}$, and draw a line λ through Q parallel to AB. Let now the plane $\bar{Q}AB$ rotate monotonously round λ until it has made a half-revolution. During this rotation the two points of intersection of γ with the rotating plane $XY\lambda$ are well determined and move continuously along γ. Hence the chord of γ joining the points of intersection X, Y of the rotating plane and γ as well as its intersection R with λ move continuously along λ. In the initial position of $XY\lambda$ this moving chord coincides with $\overline{Q}B$, while in the final position it coincides with $A\overline{Q}$. Hence the point of intersection of the moving chord and the line λ must pass (at least once) through the arbitrary interior point Q.

But, γ being a simple arc, the moving chord can pass only once through Q, because in the contrary case γ would contain four coplanar points.

In this way we have proved that through every point of the convex region there is one and only one chord of γ, i. e.

$$\omega(\gamma) = D(\gamma),$$

therefore

$$\Omega(\gamma) = \omega(\omega(\gamma)) = \omega(D(\gamma)) = D(\gamma), \qquad \text{q. e. d.}$$

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In order to determine the simple arc γ^* of a prescribed length $L = \sqrt{3}\,.\,l > 0$ whose smallest convex cover has a maximal volume, let us remark that owing to (7)

$$V = \frac{z(b) - z(a)}{6} \int_a^b [x(t)\, y(t) - y(t)\, \dot{x}(t)]\, dt = \frac{q\,.\,T}{3}.$$

where q denotes the length of the terminal chord and T is the area bounded by the orthogonal projection of γ on a plane perpendicular to the terminal chord.

We establish the following inequalities.

1. If p is the length of the normal section of the projecting cylinder of γ parallel to AB, then by the extremal property of the circle

$$T \leq \frac{p^2}{4\pi}.$$

2. If the projecting cylinder of γ parallel to AB will be developed in a plane then from the extremal property of the straight line we infer that

$$p^2 + q^2 \leq 3\, l^2.$$

3. If $l > 0$, $q > 0$ then

$$(3l^2 - q^2)\, q \equiv 2l^3 - (l - q)^2 (2l + q) \leq 2l^3.$$

Hence V satisfies the relations

$$V = \frac{q\,T}{3} \leq \frac{p^2 q}{12\pi} \leq \frac{(3l^2 - q^2)\,q}{12\pi} \leq \frac{l^3}{6\pi} = \frac{L^3}{2.3^{3/2}\pi}$$

and the equalities are valid if and only if

1. the projecting cylinder is circular, 2. the projecting cylinder being developed on a plane, γ becomes transformed into a straight line, 3. the length q of the terminating chord satisfies

$$q = l = \frac{L}{\sqrt{3}}.$$

The only space-curve satisfying these conditions is obviously an arc of a helix traced on a circular cylinder of radius $\frac{L}{\sqrt{6}\pi}$ and corresponding to a rotation 2π and a translation $\frac{L}{\sqrt{3}}$. Its smallest convex cover has the maximal value (8).

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