Publ. Math. Debrecen
47 / 1-2 (1995), 181–191

Some binomial inversions in terms of ordinary generating functions

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Abstract. In this paper we express the binomial inversion $a_n = \sum_{k=0}^{n} \binom{n}{k} b_k$ iff $b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_k$ and some related simple inversions in terms of the ordinary generating functions, and provide recent concrete examples of one of these inversion formulas from the theory of stack filters. We also emphasize the role of ordinary generating functions as a tool in finding recurrences for sequences, and give some examples involving binomial sums of Fibonacci numbers.

Introduction

For two sequences $\{a_n\}$ and $\{b_n\}$ the following simple inversion formulas are well-known:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \;\Leftrightarrow\; b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_k, \tag{0.1}$$

$$a_n = \sum_{k=n}^{m} \binom{k}{n} b_k \;\Leftrightarrow\; b_n = \sum_{k=n}^{m} \binom{k}{n} (-1)^{k-n} a_k, \tag{0.2}$$

$$a_n = \sum_{k=0}^{n} \binom{n+p}{k+p} b_k \;\Leftrightarrow\; b_n = \sum_{k=0}^{n} \binom{n+p}{k+p} (-1)^{n-k} a_k, \tag{0.3}$$

$$a_n = \sum_{k=n}^{m} \binom{k+p}{n+p} b_k \;\Leftrightarrow\; b_n = \sum_{k=n}^{m} \binom{k+p}{n+p} (-1)^{k-n} a_k, \tag{0.4}$$

Mathematics Subject Classification: 05A10, 11B37, 11B39, 94A12.

where $m \in \mathbb{N}$ or $m = \infty$. Riordan [8, Section 2.2] gave a matrix theoretic interpretation for these inversion formulas, and it is well-known that (0.1) arises naturally from the theory of exponential generating functions (see e.g. [8, Section 3.4]). The purpose of this paper is to express the present formulas in terms of the ordinary generating functions. We will also present some recent concrete examples of (0.3) from the theory of signal processing, to be more precise, from the theory of stack filters [6]. Further, we will point out that the method of ordinary generating functions makes it possible to find recurrences for binomial sums easily, and derive some examples involving binomial Fibonacci sums [1].

Throughout this paper $A(x)$ and $B(x)$ will denote the ordinary generating functions of $\{a_n\}$ and $\{b_n\}$, respectively. That is,

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,\ B(x) = \sum_{n=0}^{\infty} b_n x^n.$$

1. A characterization of (0.1)

Throughout this paper let d be an arbitrary but fixed real number.

Theorem 1. *We have*

$$a_n = \sum_{k=0}^{n} \binom{n}{k} d^{n-k} b_k \tag{1.1}$$

$$\Leftrightarrow\ A(x) = (1-dx)^{-1} B\left(x/(1-dx)\right) \tag{1.2}$$

$$\Leftrightarrow\ B(x) = (1+dx)^{-1} A\left(x/(1+dx)\right) \tag{1.3}$$

$$\Leftrightarrow\ b_n = \sum_{k=0}^{n} \binom{n}{k} (-d)^{n-k} a_k. \tag{1.4}$$

Proof. Firstly, since

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k} d^{n-k} b_k = \sum_{k=0}^{\infty} b_k x^k \sum_{n=k}^{\infty} \binom{n}{k} d^{n-k} x^{n-k}$$
$$= \sum_{k=0}^{\infty} b_k x^k (1-dx)^{-k-1} = (1-dx)^{-1} B\left(x/(1-dx)\right),$$

we have (1.1) $\Leftrightarrow$ (1.2) [3, Theorem 1].

Secondly, writing $x/(1+dx)$ for x in (1.2) we obtain (1.3), and conversely, writing $x/(1-dx)$ for x in (1.3) we obtain (1.2). Therefore (1.2) $\Leftrightarrow$ (1.3).

Thirdly, writing $-d$ for d and interchanging a_n and b_n in (1.1) we see that (1.3) $\Leftrightarrow$ (1.4). This completes the proof.

Remark. A natural representation of the inversion (1.1) $\Leftrightarrow$ (1.4) is given in terms of the exponential generating functions:

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = e^{dx} \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \ \Leftrightarrow\ \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = e^{-dx} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

For further information on exponential generating functions, see [2, Chapter 7; 7, Chapter 2].

2. A characterization of (0.2)

Theorem 2. *Let $m \in \mathbb{N}$ or $m = \infty$. Then*

$$a_n = \sum_{k=n}^{m} \binom{k}{n} d^{k-n} b_k \tag{2.1}$$

$$\Leftrightarrow\ A(x) = B(x+d) \tag{2.2}$$

$$\Leftrightarrow\ B(x) = A(x-d) \tag{2.3}$$

$$\Leftrightarrow\ b_n = \sum_{k=n}^{m} \binom{k}{n} (-d)^{n-k} a_k. \tag{2.4}$$

(If $m \in \mathbb{N}$, we assume $a_n = b_n = 0$ for $n > m$.)

Proof. Since

$$\begin{aligned}\sum_{n=0}^{m} x^n \sum_{k=n}^{m} \binom{k}{n} d^{k-n} b_k &= \sum_{k=0}^{m} b_k \sum_{n=0}^{k} \binom{k}{n} d^{k-n} x^n \\ &= \sum_{k=0}^{m} b_k (x+d)^k = B(x+d),\end{aligned}$$

we obtain (2.1) $\Leftrightarrow$ (2.2). Then writing $x-d$ for x in (2.2) and conversely writing $x+d$ for x in (2.3) we see that (2.2) $\Leftrightarrow$ (2.3). Finally, writing $-d$ for d and interchanging a_n and b_n in (2.1) we see that (2.3) $\Leftrightarrow$ (2.4). This completes the proof.

Remark. A well-known example of the inversion (2.2) $\Leftrightarrow$ (2.3) is the relationship between ordinary generating functions for probabilities and binomial moments (see e.g. [7, Section 2.6]).

3. A characterization of (0.3)

For the sums in relations (0.3) and (0.4) we define the binomial coefficients in the case of negative integers. If $n \geq 0, k < 0$ or $n < 0, k \geq 0$, we define $\binom{n}{k} = 0$, and if $n < 0, k < 0$, we define $\binom{n}{k} = (-1)^{n+k}\binom{-k-1}{-n-1}$ (see [7, p. 5]). Then it is easy to see that the recurrence relation $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ holds for all integers n and k.

Theorem 3. *Let $p \in \mathbb{Z}$. Then*

$$a_n^{(p)} = \sum_{k=0}^{n} \binom{n-p}{k-p} d^{n-k} b_k \tag{3.1}$$

$$\Leftrightarrow \quad A^{(p)}(x) = (1-dx)^{p-1} B\left(x/(1-dx)\right) \tag{3.2}$$

$$\Leftrightarrow \quad B(x) = (1+dx)^{p-1} A^{(p)}\left(x/(1+dx)\right) \tag{3.3}$$

$$\Leftrightarrow \quad b_n = \sum_{k=0}^{n} \binom{n-p}{k-p} (-d)^{n-k} a_k^{(p)}. \tag{3.4}$$

Proof. We shall first prove that (3.1) $\Leftrightarrow$ (3.2). Assume that (3.2) holds. We proceed by induction on p to prove that (3.1) holds. The result holds for $p = 0$ by Theorem 1. Assume that (3.1) holds for $p = q$ (≥ 0). By (3.2), $A^{(q+1)}(x) = (1-dx)A^{(q)}(x)$ and thus $a_0^{(q+1)} = a_0^{(q)} = b_0$ and for $n \geq 1$,

$$\begin{aligned} a_n^{(q+1)} = a_n^{(q)} - d\, a_{n-1}^{(q)} &= \sum_{k=0}^{n-1} \left[\binom{n-q}{k-q} - \binom{n-1-q}{k-q} \right] d^{n-k} b_k + b_n \\ &= \sum_{k=0}^{n} \binom{n-(q+1)}{k-(q+1)} d^{n-k} b_k. \end{aligned} \tag{3.5}$$

Since $\binom{-(q+1)}{-(q+1)} = 1$, (3.5) holds for all $n \geq 0$. For negative integers p the proof is analogous. Therefore (3.2) $\Rightarrow$ (3.1). Since the generating function is unique, we have (3.1) $\Leftrightarrow$ (3.2).

The other equivalences follow in a similar way to the analogous results in Theorem 1. This completes the proof.

4. A characterization of (0.4)

In characterizing (0.4) we need the following notation. If $F(x) = f_{-p}x^{-p} + \cdots + f_{-1}x^{-1} + \sum_{n=0}^{\infty} f_n x^n$ $(p \geq 0)$, then we denote

$$\operatorname{neg}(F(x)) = f_{-p}x^{-p} + \cdots + f_{-1}x^{-1}.$$

Theorem 4. *Let $m \in \mathbb{N}$ or $m = \infty$. If p is a nonnegative integer, then*

$$a_n^{(p)} = \sum_{k=n}^{m} \binom{k+p}{n+p} d^{k-n} b_k \tag{4.1}$$

$$\Leftrightarrow A^{(p)}(x) = (1+d/x)^p B(x+d) - \operatorname{neg}[(1+d/x)^p B(x+d)] \tag{4.2}$$

$$\Leftrightarrow B(x) = (1-d/x)^p A^{(p)}(x-d) - \operatorname{neg}[(1-d/x)^p A^{(p)}(x-d)] \tag{4.3}$$

$$\Leftrightarrow b_n = \sum_{k=n}^{m} \binom{k+p}{n+p} (-d)^{n-k} a_k^{(p)}. \tag{4.4}$$

(If $m \in \mathbb{N}$, then we assume $a_n^{(p)} = b_n = 0$ for $n > m$.)

Proof. We shall first prove that (4.1) $\Leftrightarrow$ (4.2). Assume that (4.2) holds. We proceed by induction on p to prove that (4.1) holds. If $p = 0$, the result is given in Theorem 2. Assume that (4.1) holds for $p = q$ (≥ 0). By (4.2),

$$A^{(q+1)}(x) = (1+d/x)A^{(q)}(x) - \operatorname{neg}[(1+d/x)A^{(q)}(x)]$$

and thus

$$\begin{aligned} a_n^{(q+1)} = d\, a_{n+1}^{(q)} + a_n^{(q)} &= \sum_{k=n+1}^{m} \left[\binom{k+q}{n+1+q} + \binom{k+q}{n+q} \right] d^{k-n} b_k + b_n \\ &= \sum_{k=n}^{m} \binom{k+(q+1)}{n+(q+1)} d^{k-n} b_k. \end{aligned}$$

Therefore (4.1) holds and we have thus proved that (4.2) $\Rightarrow$ (4.1). Since the generating function is unique, we have (4.1) $\Leftrightarrow$ (4.2).

The other equivalences follow in a similar way to the analogous results in Theorem 2. This completes the proof.

Theorem 5. *Let $m \in \mathbb{N}$ or $m = \infty$. If p is a negative integer, then*

$$a_n^{(p)} = \sum_{k=n}^{m} \binom{k+p}{n+p} d^{k-n} b_k \tag{4.5}$$

$$\Leftrightarrow B(x+d) = (1+d/x)^{-p} A^{(p)}(x) - \operatorname{neg}[(1+d/x)^{-p} A^{(p)}(x)] \tag{4.6}$$

$$\Leftrightarrow A^{(p)}(x-d) = (1-d/x)^{-p} B(x) - \operatorname{neg}[(1-d/x)^{-p} B(x)] \tag{4.7}$$

$$\Leftrightarrow b_n = \sum_{k=n}^{m} \binom{k+p}{n+p} (-d)^{n-k} a_k^{(p)}. \tag{4.8}$$

(If $m \in \mathbb{N}$, we assume $a_n = b_n = 0$ for $n > m$.)

Theorem 5 can be proved in a similar way to Theorem 4. We omit the details.

5. Examples of (0.3) from the theory of stack filters

Consider a stack filter (for definition, see e.g. [5, 9]). Let the inputs be i.i.d. with distribution function $\Phi(\cdot)$. It is known [6, Section 1] that the output distribution function $\Psi(\cdot)$ can be written in the following three forms:

$$\Psi(y) = \sum_{n=1}^{N} r_n \sum_{k=n}^{N} \binom{N}{k} \Phi(y)^k (1-\Phi(y))^{N-k},$$

$$\Psi(y) = \sum_{n=1}^{N} c_n \Phi(y)^n,$$

$$\Psi(y) = \sum_{n=1}^{N} a_n \Phi(y)^n (1-\Phi(y))^{N-n},$$

where the coefficients r_n, c_n and a_n have a certain natural interpretation (see [6]).

Kuosmanen et al. [6, Section 2] study the connections between the coefficients r_n, c_n and a_n. The connections are given by

$$\binom{N}{n} \sum_{k=1}^{n} r_k = a_n, \tag{5.1}$$

$$c_n = \binom{N}{n} \sum_{k=1}^{n} \binom{n-1}{k-1} (-1)^{n-k} r_k, \tag{5.2}$$

$$c_n = \sum_{k=1}^{N} \binom{N-k}{n-k} (-1)^{n+k} a_k. \tag{5.3}$$

Kuosmanen et al. [6, Section 2] express (5.1) to (5.3) in matrix forms and derive their inverse forms by inverting the appropriate matrices. Equations (5.1) to (5.3) and their inverse forms can also be interpreted in a natural way via ordinary generating functions. The ordinary generating function of $\sum_{k=1}^{n} r_k$ in (5.1) is plainly $R(x)/(1-x)$ and therefore (5.1) can be inverted by multiplication by $1-x$. We do not go into the details of (5.1) as they are not connected with binomial inversions. The main purpose of this section is to point out that equations (5.2) and (5.3) can also be written in a natural way in terms of ordinary generating functions and that the inversions of (5.2) and (5.3) are, in fact, special cases of Theorem 3.

Firstly, we write (5.2) as

$$b_n = \sum_{k=1}^{n} \binom{n-1}{k-1} (-1)^{n-k} r_k, \tag{5.4}$$

where $c_n = b_n \binom{N}{n}$. By Theorem 3, (5.4) holds if and only if

$$B(x) = R(x/(1+x))$$

if and only if

$$R(x) = B(x/(1-x))$$

if and only if

$$r_n = \sum_{k=1}^{n} \binom{n-1}{k-1} b_k. \tag{5.5}$$

This is the inverse form of (5.2).

Secondly, we write (5.3) in the form of Theorem 3 as

$$c_n = \sum_{k=1}^{N} \binom{n-(N+1)}{k-(N+1)} a_k. \tag{5.6}$$

Now, applying Theorem 3 we see that (5.6) holds if and only if

$$C(x) = (1-x)^N A(x/(1-x))$$

if and only if

$$A(x) = (1+x)^N C(x/(1+x))$$

if and only if

$$a_n = \sum_{k=1}^{n} \binom{n-(N+1)}{k-(N+1)} (-1)^{n-k} c_k.$$

The last identity can be written as

$$a_n = \sum_{k=1}^{n} \binom{N-k}{N-n} c_k. \tag{5.7}$$

This is the inverse form of (5.3).

6. Finding recurrences

Let $\{b_n\}$ be an rth order linear recurrence sequence satisfying the relation

$$b_{n+r} = c_{r-1}b_{n+r-1} + c_{r-2}b_{n+r-2} + \cdots + c_0 b_n \quad (n = 0, 1, \dots). \tag{6.1}$$

It can then be shown (cf. [4]) that $B(x)$ is of the form

$$B(x) = \frac{b_t x^t (1-\beta_1 x) \cdots (1-\beta_s x)}{(1-\alpha_1 x) \cdots (1-\alpha_r x)}, \tag{6.2}$$

where t is the least integer for which $b_t \neq 0$, and $t + s = r - 1$, and $\alpha_1, \dots, \alpha_r$ are the roots of the characteristic polynomial of the relation (6.1). If

$$a_n = \sum_{k=0}^{n} \binom{n}{k} d^{n-k} b_k, \tag{6.3}$$

then, by Theorem 1, we obtain after some algebraic manipulations that

$$\begin{aligned} A(x) = (1-dx)^{-1} B\left(\frac{x}{1-dx}\right) = \\ = \frac{b_t x^t (1-(\beta_1+d)x) \cdots (1-(\beta_s+d)x)}{(1-(\alpha_1+d)x) \cdots (1-(\alpha_r+d)x)}. \end{aligned} \tag{6.4}$$

We can thus deduce (cf. [4]) that $\{a_n\}$ satisfies an rth order linear recurrence relation, whose characteristic polynomial has roots $\alpha_1+d, \dots, \alpha_r+d$.

More generally, if

$$a_n = \sum_{k=0}^{n} \binom{n-p}{k-p} d^{n-k} b_k \tag{6.5}$$

and $B(x)$ is as in (6.2), then, by Theorem 3, we obtain after some algebraic manipulations that

$$A(x) = (1-dx)^p \frac{b_t x^t (1-(\beta_1+d)x)\cdots(1-(\beta_s+d)x)}{(1-(\alpha_1+d)x)\cdots(1-(\alpha_r+d)x)}. \tag{6.6}$$

Therefore $\{a_n\}$ satisfies a linear recurrence relation of order $r+|p|$. If $p>0$, then the roots of the characteristic polynomial of the relation are $\alpha_1+d,\dots,\alpha_r+d,0,\dots,0$ (0, p times). (Note that, in addition to the p zeros, some of the roots $\alpha_1+d,\dots,\alpha_r+d$ may be equal to zero.) If $p<0$, then the roots of the characteristic polynomial of the relation are $\alpha_1+d,\dots,\alpha_r+d,d,\dots,d$ (d, $-p$ times). (Also some of the roots $\alpha_1+d,\dots,\alpha_r+d$ may be equal to d.)

If $a_n=\sum_{k=n}^{\infty}\binom{k}{n}d^{n-k}b_k$ and $B(x)$ is as in (6.2), then it can be seen after some algebraic manipulations that $\{a_n\}$ satisfies a linear recurrence relation of order r, whose characteristic polynomial has roots $\alpha_1/(1-\alpha_1 d),\dots,\alpha_r/(1-\alpha_r d)$, provided $1-\alpha_i d\neq 0 \;\; (i=1,\dots,r)$.

7. Examples involving binomial sums of Fibonacci numbers

In this section we present some examples in which we apply (6.4) and (6.6) to obtain expressions for certain binomial sums of Fibonacci numbers.

Example 1. Let $b_n=F_n$, the nth Fibonacci number, and $d=1$ in (6.3). Then

$$a_n=\sum_{k=0}^{n}\binom{n}{k}F_k,$$

and, by (6.4),

$$A(x)=\frac{x}{(1-\alpha^2x)(1-\beta^2x)}=\frac{x}{1-3x+x^2}, \tag{7.1}$$

where $\alpha=(1+\sqrt{5})/2$, $\beta=(1-\sqrt{5})/2$. The first expression of $A(x)$ shows that $a_n=C_1\alpha^{2n}+C_2\beta^{2n}$ for some constants C_1 and C_2. Solving C_1 and C_2 gives $a_n=(\alpha^{2n}-\beta^{2n})/\sqrt{5}$. Application of the Binet form $F_n=(\alpha^n-\beta^n)/(\alpha-\beta)=(\alpha^n-\beta^n)/\sqrt{5}$ gives

$$a_n=F_{2n}, \quad n\ge 0 \tag{7.2}$$

(cf. [1, equation (24)]). Further, the second expression of $A(x)$ in (7.1) yields the recurrence

$$a_{n+2}=3a_{n+1}-a_n, \quad n\ge 0.$$

Note that (7.2) could also be derived easily with aid of exponential generating functions. We do not present the details here.

Example 2. Let $b_n = F_n$, $d = 1$ and $p = -1$ in (6.5). Then

$$a_n = \sum_{k=0}^{n} \binom{n+1}{k+1} F_k,$$

and, by (6.6),

$$A(x) = \frac{x}{(1-x)(1-\alpha^2 x)(1-\beta^2 x)} = \frac{x}{1-4x+4x^2-x^3}\ . \tag{7.3}$$

The first expression of $A(x)$ shows that $a_n = C_1 + C_2\alpha^{2n} + C_3\beta^{2n}$ for some constants C_1, C_2 and C_3. Solving C_1, C_2 and C_3 gives $a_n = -1 + \left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)\alpha^{2n} + \left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right)\beta^{2n}$. Application of the Binet forms $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$, where L_n is the nth Lucas number, gives

$$a_n = -1 + L_{2n}/2 + F_{2n}/2, \quad n \geq 0. \tag{7.4}$$

Further, the second expression of $A(x)$ in (7.3) yields the recurrence

$$a_{n+3} = 4a_{n+2} - 4a_{n+1} + a_n, \quad n \geq 0.$$

Example 3. Let $b_n = F_n$, $d = 1$ and $p = 1$ in (6.5). Then

$$a_n = \sum_{k=0}^{n} \binom{n-1}{k-1} F_k,$$

and, by (6.6),

$$A(x) = \frac{x(1-x)}{(1-\alpha^2 x)(1-\beta^2 x)} = -1 + \frac{1-2x}{1-3x+x^2}\ . \tag{7.5}$$

This shows that $a_n = -1 \cdot 0^n + C_1\alpha^{2n} + C_2\beta^{2n}$ for some constants C_1 and C_2, where $0^0 = 1$. Solving C_1 and C_2 gives $a_n = -1 \cdot 0^n + \left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right)\alpha^{2n} + \left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)\beta^{2n}$. Application of the Binet forms of F_n and L_n gives

$$a_n = -1 \cdot 0^n + L_{2n}/2 - F_{2n}/2, \quad n \geq 0. \tag{7.6}$$

Further, the second expression of $A(x)$ in (7.5) yields the recurrence

$$a_{n+3} = 3a_{n+2} - a_{n+1}, \quad n \geq 0.$$

Remark. Further examples involving binomial sums of Fibonacci numbers could be derived in a similar way. For instance, Examples 1–3 with $d = -1$.

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(Received September 2, 1994; revised November 14, 1994)