Remark on a theorem of Fejér.

By P. TURÁN in Budapest.

1. The investigation of the trigonometrical and power-series with monotonic coefficients — a subject starting with ABEL and SCHLOEMILCH — made a great progress recently mainly by the papers of FEJÉR and SZEGÖ. Here we mean also the case when the coefficients

(1. 1) $$a_0, a_1, \ldots, a_n, \ldots$$

form a sequence of *higher* monotonicity; the sequence (1. 1) is called, as well-known, monotonic of order k if

(1. 2) $$a_n - \binom{\nu}{1} a_{n+1} + \binom{\nu}{2} a_{n+2} - \ldots + (-1)^{\nu} \binom{\nu}{\nu} a_{n+\nu} \geqq 0$$
$$(\nu = 0, 1, \ldots, k;\ n = 0, 1, \ldots).$$

These investigations are based on interesting properties of the partial-sums

(1. 3) $$s_n(z) = s_n^{(0)}(z) = 1 + z + z^2 + \ldots + z^n$$

of the geometrical series as well as of the iterated partial-sums $s_n^{(k)}(z)$, where these $s_n^{(k)}(z)$ polynomials are defined recursively by

(1. 4) $$s_n^{(k)}(z) = s_0^{(k-1)}(z) + s_1^{(k-1)}(z) + \ldots + s_n^{(k-1)}(z)\ (k = 1, 2, \ldots;\ n = 0, 1, 2, \ldots).$$

These properties refer partly to the behaviour on the unitcircle, partly on the diameter of it, lying on the real axis. One of this properties is the following: [1])

Given a non-negative integer μ, the polynomials

(1. 5) $$s_n^{(\mu)}(z) \qquad (n = 0, 1, \ldots)$$

together with their first μ derivatives (if they are not identically 0) are positive on the segment $-1 < x < +1$.

From (1. 4) it follows as Fejér l. c. remarked, that for the proof of this theorem it is sufficient to show it for the sequence of the μ^{th} derivatives only; i. e.

(1. 6) $$A_n(x, \mu) = \frac{d^{\mu}}{dx^{\mu}} s_n^{(\mu)}(x) > 0 \qquad (n = \mu, \mu + 1, \ldots;\ -1 < x < 1).$$

This means that these $A_n(x, \mu)$ polynomials have no zeros on the segment $-1 < x < 1$. In what follows I shall show by a slight modification of FEJÉR's first proof for (1. 6) that more exact information about this zeros can be derived from the following

1) L. FEJÉR: Untersuchungen über Potenzreihen mit mehrfach monotoner Koeffizientenfolge. *Acta Sci Math. Szeged* **8** (1936), p. 88—115.

Theorem 1. *All the zeros of the polynomials* $\frac{d^\mu}{dx^\mu} s_n^{(\mu)}(x)$ $(n=\mu+1, \mu+2, \ldots)$ *lie on the periphery of the unit-circle.*

For $\mu=0$ this asserts the well-known fact that the zeros of the polynomials $1+z+z^2+\ldots+z^n$ lie on the unit-circle. For $\mu=1$ this theorem was proved by another method by EGERVÁRY [2]). From this theorem 1 it follows that the sign of the polynomials $A_n(x, \mu)$ is constant on the segment $-1 \leqq x \leqq +1$; since its coefficients are evidently non-negative, the non-negativity of the $A_n(x, \mu)$-s, i. e. the assertion (1. 6) follows immediately. But we can show a little more. We can show that the polynomials

$$A_n(x, \mu) = \frac{d^\mu}{dx^\mu} s_n^{(\mu)}(x)$$

are non-negative on the whole real axis if n is even; if n is odd then the only zero on the real axis is at $x=-1$ and is simple. Summing up these two assertions (which are interesting of course only for negative x-values) in one we can state it as

Theorem 2. *We have on the whole real axis for any fixed non-negative integer* μ

$$\frac{1}{1+x^n} \cdot \frac{d^\mu}{dx^\mu} s_{n+\mu}^{(\mu)}(x) > 0, \quad (n=0, 1, \ldots).$$

2. In the proof an important role is played by the ultraspherical polynomials [3]) $P_n^{(j)}(\zeta)$. These are defined by the generatorfunction

$$\sum_{n=0}^{\infty} P_n^{(j)}(\zeta) w^n = \frac{1}{(1-2\zeta w+w^2)^j}. \tag{2. 1}$$

If $j=\frac{1}{2}$, these $P_n^{(j)}(\zeta)$-s are the LEGENDRE-polynomials, if $j=1$ then the corresponding polynomials are the TCHEBICHEFF-polynomials of second kind. They are even or odd polynomials according to the parity on n. It is well-known [4]) that all the zeros of the polynomials $P_n^{(j)}(\zeta)$ are real, simple and lie on the segment $-1<\zeta<1$ if e. g. $j>0$; hence they can be written in the form

$$\zeta_\nu = \cos\vartheta_\nu,\ 0<\vartheta_1<\vartheta_2<\ldots<\vartheta_n<\pi,\ \nu=1, 2, \ldots, n. \tag{2. 2}$$

The idea of the proof of theorem 1 is representing the polynomials $A_n(x,\mu)$ by ultraspherical polynomials. Another instance of appearence of ultraspherical polynomials in the theory of these $s_n^{(\mu)}(z)$ polynomials will be given in a forthcoming paper of FEJÉR and SZEGŐ [5]); in this paper they prove e. g. another theorem of EGERVÁRY [2]), according which the polynomials

---

[2]) E. EGERVÁRY: Abbildungseigenschaften der arithmetischen Mittel der geometrischen Reihe. *Math. Z.* **42** (1937), p. 221—230.

[3]) The classical theory of these polynomials is given e. g. in SZEGŐ's book: Orthogonal polynomials. *Amer. Math. Soc. Coll. Publ.* Vol. XXIII. (Newyork, 1939.)

[4]) See SZEGŐ's book p. 113.

[5]) On special conformal mappings. (To be printed in *Duke Math. Journ.*)

$$s_n^{(3)}(e^{i\vartheta}) = u(\vartheta) + iv(\vartheta)$$

give *convex* curves as map of the unit-circle by expressing the relevant expression

$$u'(\vartheta)v''(\vartheta) - v'(\vartheta)u''(\vartheta)$$

as a square of ultraspherical polynomials.

3. To obtain the above mentioned representation of $A_n(x,\mu)$ we start from FEJÉR's generator-representation

$$\frac{1}{\mu!}\sum_{n=0}^{\infty} A_{n+\mu}(x,\mu)z^n = \frac{1}{((1-z)(1-xz))^{\mu+1}}, \tag{3.1}$$

valid for $|z| < \min\left(1, \frac{1}{|x|}\right)$. We suppose first $x \neq 0$. Then replacing x by $e^{i\Theta}$, z by $ze^{-i\frac{\Theta}{2}}$, we obtain

$$(1-z)(1-xz) = (1-ze^{-i\frac{\Theta}{2}})(1-ze^{i\frac{\Theta}{2}}) = 1-2z\cos\frac{\Theta}{2}+z^2$$

i. e. for all sufficiently small $|z|$

$$\frac{1}{\mu!}\sum_{n=0}^{\infty} A_{n+\mu}(e^{i\Theta},\mu)\,e^{-ni\frac{\Theta}{2}}z^n = \frac{1}{(1-2z\cos\frac{\Theta}{2}+z^2)^{\mu+1}}.$$

Comparing this with (2. 1) we obtain the required representation

$$A_{n+\mu}(e^{i\Theta},\mu) = \frac{d^\mu}{dx^\mu}s_{n+\mu}^{(\mu)}(x)_{x=e^{i\Theta}} = \mu!\,e^{ni\frac{\Theta}{2}}P_n^{(\mu+1)}\left(\cos\frac{\Theta}{2}\right) \tag{3.2}$$

or also

$$A_{n+\mu}(x,\mu) = \frac{d^\mu}{dx^\mu}s_{n+\mu}^{(\mu)}(x) = \mu!\,x^{\frac{n}{2}}P_n^{(\mu+1)}\left(\frac{\sqrt{x}+\frac{1}{\sqrt{x}}}{2}\right). \tag{3.3}$$

If $x=0$, this has a sense only interpreting the right as limit for $x \to 0$.

To deduce theorem 1 from the representation (3. 2) we have to remark only from (2. 2) and (3. 2) that $A_{n+\mu}(e^{i\Theta},\mu)$ vanishes for

$$\Theta = 2\vartheta_\nu \qquad (\nu = 1, 2, \ldots, n)$$

i. e. $A_{n+\mu}(x,\mu)$ for

$$x = e^{2i\vartheta_\nu} \qquad (\nu = 1, 2, \ldots, n) \tag{3.4}$$

which give n different values; since $A_{n+\mu}(x,\mu)$ is a polynomial of degree n, the values in (3. 4) are *all* the zeros.

To prove our theorem 2 we have only to ask when is one of the x_ν's real. Owing to $0 < \vartheta_\nu < \pi$ only $x_\nu = -1$ can occur. In this case $\vartheta_\nu = \frac{\pi}{2}$, i. e. $P_n^{(\mu+1)}(0) = 0$. According to **2** this occurs if and only if n is odd and $x=-1$ is then a simple zero of $A_{n+\mu}(x,\mu)$. Q. e. d.

(Received October 8, 1949.)