On some linear functional equations.

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§ 1. Introduction.

In a recent paper[1]) we considered the asymptotic behaviour, when $x \to \infty$, of the solutions of equations of the type

$$(1.1) \qquad w(x)\, f'(x) + f(x) - f(x-1) = 0,$$

$w(x)$ a given continuous function, $w(x) > 0$. We proved that any solution is bounded for $x \to \infty$, and, if $w(x)$, $w'(x)$, $w''(x)$ tend to 0 sufficiently rapidly, that any solution tends to a periodic function with period 1. For instance, all solutions of

$$(1.2) \qquad x^{-\alpha} f'(x) + f(x) - f(x-1) = 0$$

are asymptotically periodic, if α is a constant > 1. If $\frac{1}{2} < \alpha \leqq 1$, the behaviour is similar but slightly more complicated.

Here we prove, that if $w(x)$ is not too small, then any solution of (1.1) tends to a constant. A sufficient condition is

$$(1.3) \qquad w(x) \geqq \frac{1}{\log x} \quad \text{for} \quad x \to \infty.$$

It is very easy to prove this result if we suppose that $w(x) \geqq c$, where c is a constant > 1. We then have

$$|f'(x)| \leq \frac{1}{c} \operatorname*{Max}_{x-1 \leqq t \leqq x} |f'(t)|,$$

and hence $f'(x) = O(c^{-x})$. An example for this simple case is the equation (1.2) for $\alpha < 0$.

As applications we shall consider some equations of the form $F'(x) = K(x)\, F(x-1)$ ($K(x)$ a given continuous and positive funtion), which can be transformed into the form (1.1) Previously[2]) the examples $K(x) = e^x$

[1]) N. G. DE BRUIJN: The asymptotically periodic behaviour of the solutions of some linear functional equations. *Amer. J. Math.* 71 (1949), 313–330. In the sequel this paper is quoted as A. P.

[2]) A. P. section 6.

and $K(x) = \exp(x^2)$ were shown to lead to the case of asymptotic periodicity. In the present paper we consider some cases where the order of magnitude of $K(x)$ is considerably lower.

In view of these applications we consider the equation (2. 5) instead of (1. 1). The result can even be extended to

$$w(x)f'(x)+p(x)f(x)-q(x)f(x-1)=r(x),$$

where $p(x)=1+O(\Phi)$, $q(x)=1+O(\Phi)$, $r(x)=O(\Phi)$, and Φ satisfies (2. 1).

The content of the present paper is independent from the one quoted before.

§ 2. The convergence theorem.

Let $\Phi(x)$ be a function with

$$\Phi(x)\downarrow 0 \quad \text{for } x\to\infty, \qquad \int_0^\infty \Phi(x)\,dx<\infty, \tag{2. 1}$$

and let the continuous functions $w(x)$ and $q(x)$ satisfy, for $x\geqq 0$,

$$q(x)=1+O\{\Phi(x)\}, \tag{2. 2}$$

$$w(x)>0, \tag{2. 3}$$

$$\sum_{n=1}^{\infty} \exp\left\{-\int_{n-1}^{n}\frac{dt}{w(t)}\right\}=\infty. \tag{2. 4}$$

We shall prove:

If, for $x\geqq 0$, $f(x)$ *satisfies*

$$w(x)\,f'(x)+f(x)-q(x)\,f(x-1)=0, \tag{2. 5}$$

then $\lim\limits_{x\to\infty} f(x)$ *exists.*

Proof. Suppose $x\geqq n\geqq 1$. By considering (2. 5) as a linear differential equation, we can express $f(x)$ in terms of $q(x)f(x-1)$.

Putting

$$W(\alpha,\beta)=\exp\left\{-\int_{\alpha}^{\beta}\frac{dt}{w(t)}\right\} \tag{2. 6}$$

we find

$$f(x)=W(n,x)\,f(n)+\int_n^x \frac{q(t)f(t-1)}{w(t)}\,W(t,x)\,dt. \tag{2. 7}$$

We have $\int_n^x \frac{W(t,x)}{w(t)}\,dt=1-W(n,x)$, and hence, for some η $(n\leqq\eta\leqq x)$

$$f(x)=W(n,x)\,f(n)+q(\eta)\,f(\eta-1)\,\{1-W(n,x)\}. \tag{2. 8}$$

Now put

$$M_n=\max_{n-1\leqq x\leqq n}|f(x)|,\quad \Omega_n=\max_{n-1\leqq x\leqq n} f(x),\quad \omega_n=\min_{n-1\leqq x\leqq n} f(x).$$

We have $|q(\eta)-1|<C\Phi(n)$, where C is an absolute constant. It follows from (2. 8) that, for $n\leqq x\leqq n+1$

$$|f(x)|\leqq W(n,x)\,M_n+\{1+C\Phi(n)\}\,M_n\{1-W(n,x)\}\leqq M_n\{1+C\Phi(n)\},$$

and so

$$M_{n+1}\leqq M_n\{1+C\Phi(n)\}.$$

Now (2. 1) shows that M_n is bounded[3]), $M_n\leqq M_1B$, where B depends on Φ and C only. Put $\max\limits_{0\leqq x<\infty}|f(x)|=M$.

Again take $n\leqq x\leqq n+1$ in (2. 8); we infer

$$f(x)\leqq W(n,x)\;f(n)+\Omega_n\{1-W(n,x)\}+MC\Phi(n).$$

We have $1\geqq W(n,x)\geqq W(n,n+1)$, and $f(n)\leqq\Omega_n$. Hence, for $n\leqq x\leqq n+1$,

$$f(x)\leqq\Omega_n-W(n,x)\;\{\Omega_n-f(n)\}+MC\Phi(n)\leqq$$
$$\leqq\Omega_n-W(n,n+1)\;\{\Omega_n-f(n)\}+MC\Phi(n).$$

Consequently

$$(2.\,9)\qquad \Omega_{n+1}\leqq\{1-W(n,n+1)\}\;\Omega_n+W(n,n+1)\;f(n)+MC\Phi(n).$$

Analogously

$$\omega_{n+1}\geqq\{1-W(n,n+1)\}\,\omega_n+W(n,n+1)\;f(n)-MC\Phi(n-1).$$

Hence $\delta_n=\Omega_n-\omega_n$ satisfies

$$(2.\,10)\qquad \delta_n\geqq 0,\ \delta_{n+1}\leqq\{1-W(n,n+1)\}\;\delta_n+2\,MC\Phi(n).$$

Since

$$\prod_{n=1}^{\infty}\{1-W(n,n+1)\}=0,\ \sum_{n=1}^{\infty}\Phi(n)<\infty$$

we easily infer that $\delta_n\to 0$ for $n\to\infty$. It remains to be proved that $\lim\Omega_n$ exists.

From (2. 9) we infer

$$\Omega_{n+1}\leqq\Omega_n+MC\Phi(n)$$

and so

$$(2.\,11)\qquad \Omega_{n+k}\leqq\Omega_n+\sum_{\nu=n}^{\infty}MC\Phi(\nu)\qquad(k=1,2,3,\ldots).$$

It follows that

$$\limsup\,\Omega_n\leq\liminf\,\Omega_n+0,$$

where both sides exist, since Ω_n is bounded. Consequently $\lim\,\Omega_n$ exists. This proves the theorem.

§ 3. Estimation of $f(x)-\lim f(x)$.

Put

$$\lim_{x\to\infty}f(x)=\lim_{n\to\infty}\Omega_n=\lim_{n\to\infty}\omega_n=A.$$

Formula (2. 11) gives

$$\Omega_n\geqq A-\sum_{n}^{\infty}MC\Phi(\nu).$$

[3]) An alternative proof was given in A. P., Theorem 1.

Analogously we find

$$\omega_n \leqq A + \sum_{\nu=n}^{\infty} MC\Phi(\nu)$$

and consequently, since $\Omega_n - \delta_n \leqq f(x) \leqq \omega_n + \delta_n \qquad (n-1 \leqq x \leqq n)$,

$$(3.\,1) \qquad |f(x)-A| \leqq \delta_{n+1} + \sum_{\nu=n+1}^{\infty} MC\Phi(\nu) \qquad (n \leqq x \leqq n+1).$$

δ_n can be estimated from (2. 10) by iteration; we obtain

$$(3.\,2) \qquad \delta_{n+1} \leqq \delta_1 \prod_{\nu=1}^{n} \{1 - W(\nu\ \nu+1)\} + 2MC \sum_{\nu=1}^{n} \Phi(\nu) \prod_{j=\nu+1}^{n} \{1 - W(j\ j+1)\}.$$

From (3. 1) and (3. 2) an estimate for $|f(x) - A|$ can be deduced whenever $w(x)$ and $\Phi(x)$ are known explicitly.

§ 4. Examples.

Condition (2. 4) is easily seen to be satisfied if $w(x) \geqq \frac{1}{\log x}$. Probably the convergence theorem remains true if (2. 4) is replaced by $w(x) > \frac{a}{\log x}$ whenever a is a positive constant. On the other hand, of $w(x) = x^{-\alpha}$, $\alpha > \frac{1}{2}$, oscillating solutions of (2. 5) do appear[4]); if $\alpha > 1$ we even know[5]) that lim $f(x)$ exists if and only if $f(x)$ is a constant.

We will now give some examples of equations of the form

$$(4.\,1) \qquad G'(x) = K(x)\ G(x-1) \qquad (K(x) > 0).$$

Example 1.

$$(4.\,2) \qquad G'(x) = x^{-1}\ G(x-1) \qquad (x > 0).$$

A special solution is $G_0(x) = x + 1$. Putting
$G(x) = (x+1)\ f(x)$ we obtain $(x+1)\ f'(x) + f(x) - f(x-1) = 0$.
Hence lim $f(x)$ exists for any solution $f(x)$. We have

$$W(n, n+1) = \exp\left\{-\int_{n}^{n+1} \frac{dx}{x+1}\right\} = \frac{n+1}{n+2},$$

and hence, by (3. 2) $\delta_{n+1} \leqq C_1/(n+2)!$. It follows that for any solution of (4. 1) we have

$$G(x) = Ax + A + O\{([x+1]!)^{-1}\}.$$

It is not difficult to obtain an error term $O\{$[illegible]$(x+2)\}$ by means of a small modification of our argument.

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4) A. P. section 4.

5) A. P. section 5.

Example 2. We can also determine the asymptotic behaviour of the solutions of (4. 1) in cases where it is impossible to find special solutions. To this end we try[6]) to find a function $h(x)$ such that for the function

$$\eta(x) = \log K(x) - \int_0^1 h(x-t)\, dt - \log h(x) \tag{4. 3}$$

we have $\eta(x) = O\{\Phi(x)\}$, where Φ is a function satisfying (2. 1). Then, if $H'(x) = h(x)$, the substitution $G(x) = f(x) \exp H(x)$ transforms (4. 1) into

$$f'(x)/h(x) + f(x) - e^{\eta(x)} f(x-1) = 0.$$

A function $h(x)$ can often be found by iteration. Suppose, for instance that $K(x)$ is large for $x \to \infty$. Now starting with $h_0(x)$, suitably chosen, evaluating the corresponding function $\eta_0(x)$ by (4. 3), and then taking $h_1(x) = h_0(x) + \eta_0(x)$ etc., it often turns out that for some finite number k we have $\eta_k = O(\Phi)$.

We consider a rather general case. Let $K(x)$ satisfy

$$\lim_{x\to\infty} K(x) = \infty, \left(\frac{K'}{K}\right)^2 = O(\Phi),\ \frac{K''}{K} = O(\Phi) \tag{4. 4}$$

and let $\lambda(x)$ be the positive solution of

$$\lambda(x)\ e^{\lambda(x)} = K(x). \tag{4. 5}$$

The iteration process can be carried out succesfully by taking $h_0(x) = \lambda(x)$[7]). We only give the final result, which can be verified directly. If we put

$$G_0^*(x) = \exp\left\{\int_1^x \lambda(t)dt + \frac{1}{2}\lambda(x) - \frac{1}{2} \log\ (1+\lambda(x))\right\} \tag{4. 6}$$

the substitution $G(x) = G_0^*(x) f(x)$ transforms (4. 1) into an equation of the form (2. 5) with $q(x) = 1 + O\{\Phi(x-1)\}$, $w(x) = \{\lambda(x) + \frac{1}{2}\lambda'(x) - \frac{1}{2}\lambda'(x)/(1+\lambda(x))\}^{-1}$.

It is now possible to examine whether the convergence theorem can be applied. This is the case, for instance, if $\lambda'(x) > 0$, $\lambda(x) \leq \log x$, that is, $K(x)$ satisfies apart from (4. 4), the inequalities $K'(x) > 0$, $K(x) < x \log x$. Application of § 3 will now furnish the order of the error term.

We state the result for the equation

$$G'(x) = xG(x-1)$$

Any solution is of the form

$$G(x) = \left\{ A + O\left(\frac{1}{x}\right)\right\} \frac{e^{x\left(\lambda - 1 + \frac{1}{\lambda}\right) + \frac{1}{2}\lambda}}{\sqrt{\lambda+1}} \qquad (\lambda e^{\lambda} = x).$$

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[6]) cf. A. P. p. 328.

[7]) If we take $h_0(x) = \log K(x)$ the process does not terminate.

Although (4.6) was introduced under the assumption that $K(x) \to \infty$ for $x \to \infty$, it can also be used for $K(x)$ bounded. If we only suppose

$$K(x) = O(1),\ K'(x) = O(\Phi^{\frac{1}{2}}),\ K''(x) = O(\Phi)$$

we can prove that any solution of (4.1) is of the form

$$G(x) = G_0^*(x)\ \{A + o(1)\}.$$

Example 3. $\qquad G'(x) = e^{-x} G(x-1).$

It is easy to show that any solution tends to a constant for $x \to \infty$. By a suitable substitution, however, we can obtain a more precise result. Put

$$G_N(x) = \sum_{\nu=0}^{N} \frac{(-1)^\nu}{\nu!} e^{\frac{1}{2}\nu(\nu-1) - \nu x}$$

It is easily verified that the substitution $G(x) = G_N(x) f(x)$ gives an equation of the type (2.5) with $w(x) \sim e^x$, $q(x) = 1 + O(e^{-Nx})$. The estimates of § 3 now give $f(x) = A + O(e^{-Nx})$. Hence any solution of $G'(x) = e^{-x} G(x-1)$ has the divergent asymptotic expansion

$$G(x) \sim A \sum_{0}^{\infty} \frac{(-1)^\nu}{\nu!} e^{\frac{1}{2}\nu(\nu-1) - \nu x}$$

Still better results can be obtained by using the special solution

$$G_0(x) = \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}(x+t)^2 - e^{t - \frac{1}{2}} \right\} dt.$$

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