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y-Berwald spaces of dimension two and associated heterochronic systems

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Abstract. This paper classifies all two-dimensional y-Berwald spaces. Such Finsler geometries arise in time-sequencing change models in the evolution of colonial organisms.

0. Introduction

Let $N(p,q)$ denote the fundamental function of a two-dimensional Minkowski space, $g_{ij}(p,q)$ the metric tensor and let $\psi(x,y,p,q)$ denote a fixed function (positively) homogeneous of degree one in p,q (i.e. $\dot{x},\dot{y}$). In the biological works about modelling colonial animals with two morphotypes or castes [5], [6] one defines the *Associated Heterochronic System* (AHS) to be the dynamical system

$$\frac{d^2x^i}{ds^2} + (\delta^i_j\psi_k + \delta^i_k\psi_j)\,\frac{dx^j}{ds}\,\frac{dx^k}{ds} = g^{ij}\dot{\partial}_j\psi,$$

where $x^1 = x, \quad x^2 = y$, are log-biomass variables for each type $\psi_k = \dot{\partial}_k\psi$ and $ds = (g_{ij}\dot{x}^i\dot{x}^j)^{1/2}$ measures total size increment. The left-hand side represents the *time-sequencing change,* along the straight-line growth curves which are geodesics for the metric function $N(p,q)$, and is in fact a projective parameter change of the original geodesics [2], [7], [8]. The right-hand side expresses the vertical gradient influence (i.e. external and environmental) which causes a colonial organism to change the internal ecology of its two member castes (i.e. subpopulations) through the hormonal alteration of its genetically defined program of growth and differentiation [5], [6].

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In the present paper, we address the important question of when an AHS is identical to the geodesic spray of a Finsler metric function F defined only in terms of ψ and $N(p,q)$. In the case, $\psi = a_1p + a_2q$, a_i constant, it is known that $F(x,y,p,q) = e^{a_1x+a_2y} \cdot N(p,q)$, [1], [2], [3], [4]. The main result proved here is that this is the only possibility given that ψ is independent of x and y. In this instance, the AHS must be the geodesic spray of a *y-Berwald space,* which is a Finsler space with the Berwald connection coefficients independent of x and y (these would be *adapted coordinates* in such a space).

Our theorem is proved in Section 1, some difficult examples are provided in the second section. The AHS is a little understood dynamics. We hope the present work and the references will be helpful to readers interested in further discussions.

1. Two-dimensional y-Berwald spaces

We consider an n-dimensional Finsler space F^n with the fundamental function $L(x;y)$. If we put

$$L_i = \partial L/\partial x^i, \quad L_{(i)} = \partial L/\partial y^i,$$

we have for the Berwald connection (G^i_j, G^i_{jk})

$$L_i = L_{(r)}G^r_i. \tag{1.1}$$

Then we get

$$L_iy^i := L_0 = 2L_{(r)}G^r, \quad (2G^r = G^r_iy^i). \tag{1.2}$$

Next (1.1) gives

$$L_{i(j)} - L_{j(i)} = L_{(r)(j)}G^r_i - L_{(r)(i)}G^r_j. \tag{1.3}$$

We shall restrict our discussion to the *two-dimensional* case only. Then (1.3) gives the single equation

$$L_{1(2)} - L_{2(1)} := M = L_{(1)(2)}G^1_1 + L_{(2)(2)}G^2_1 - L_{(1)(1)}G^1_2 - L_{(2)(1)}G^2_2.$$

Using the notation $(p,q) = (y^1, y^2)$ and the Weierstrass invariant [2]

$$W = L_{pp}/q^2 = -L_{pq}/pq = L_{qq}/p^2,$$

the above is written in the form

$$M = -2G^1Wq + 2G^2Wp, \tag{1.4}$$

on account of $2G^r = G^r_i y^i$. Then, (1.4) together with (1.2) leads to

$$(1.5) \qquad 2G^1 L = L_0 p - L_q M/W, \quad 2G^2 L = L_0 q + L_p M/W.$$

Definition. A Finsler space is called *y-Berwald,* if there exists a covering by coordinate neighborhoods in each of which the Berwald connection coefficients G^i_{jk} are functions of y^i alone.

From the well-known equations

$$G^i_j = G^i_{jk} y^k, \quad 2G^i = G^i_j y^j, \quad G^i_j = \partial G^i/\partial y^j, \quad G^i_{jk} = \partial G^i_j/\partial y^k,$$

it is obvious that G^i_{jk} in Definition can be changed to G^i or G^i_j.

We shall deal with a *y-Berwald space of two dimensions.* Then (1.1) yields $L_{ij} = L_{j(r)} G^r_i$ and

$$(1.6) \qquad L_{j(r)} G^r_i - L_{i(r)} G^r_j = 0.$$

Since (1.1) gives

$$L_{j(r)} = \big(L_{(s)} G^s_j\big)_{(r)} = L_{(s)(r)} G^s_j + L_{(s)} G^s_{jr},$$

(1.6) is written in the form

$$L_{(s)}\big(G^s_{jr} G^r_i - G^s_{ir} G^r_j\big) = 0,$$

which is only the single equation

$$(1.7) \qquad L_p H^1 + L_q H^2 = 0,$$

where we put

$$(1.7\text{a}) \qquad H^s(p,q) = G^s_{1r} G^r_2 - G^s_{2r} G^r_1.$$

Then, (1.7) together with $L_p p + L_q q = L$ yields

$$(1.8) \qquad L_p/L = K_1, \quad L_q/L = K_2,$$

where we put

$$(1.8\text{a}) \qquad K_1(p,q) = H^2/(pH^2 - qH^1), \quad K_2(p,q) = -H^1/(pH^2 - qH^1).$$

For K_i we get

$$(K_1)_q = \{H^1 H^2 + q(H^2 H^1_q - H^1 H^2_q)\}/(pH^2 - qH^1)^2,$$

$$(K_2)_p = \{H^1 H^2 - p(H^2 H^1_p - H^1 H^2_p)\}/(pH^2 - qH^1)^2.$$

Thus we have $(K_1)_q = (K_2)_p$, i.e., K_i is a gradient vector field in the (p,q)-space, on account of the homogeneity of H^s. Thus we have a function

$K(p,q)$ satisfying $K_1 = K_p$ and $K_2 = K_q$. Thus (1.8) can be integrated to obtain

$$L(x,y;p,q) = e^{f(x,y)} N(p,q), \tag{1.9}$$

where $f(x,y)$ is some function of (x,y) and $N = e^{K(p,q)}$.

Consequently, the space under consideration must be conformal to a locally Minkowski space M^2 with the fundamental function N.

Now we shall return to (1.1); on account of (1.9) it is written as

$$f_x = (N_p G_1^1 + N_q G_1^2)/N, \quad f_y = (N_p G_2^1 + N_q G_2^2)/N.$$

The left-hand sides of these equations are functions of (x,y), while the right-hand sides are functions of (p,q) from our assumption. Hence these must be constant: $f_x = a_1$, $f_y = a_2$,
so that

$$f(x,y) = a_1 x + a_2 y + a. \tag{1.10}$$

Then the formula (1.5) yields

$$\begin{aligned} 2G^1 &= p(a_1 p + a_2 q) - (a_1 N_q - a_2 N_p) N_q / wN, \\ 2G^2 &= q(a_1 p + a_2 q) + (a_1 N_q - a_2 N_p) N_p / wN, \end{aligned} \tag{1.11}$$

where w is the Weierstrass invariant of M^2 :

$$w = N_{pp}/q^2 = -N_{pq}/pq = N_{qq}/p^2. \tag{1.12}$$

Consequently, G^i are functions of (p,q) alone and the space is y-Berwald.

Theorem. *Any y-Berwald space of dimension two is conformal to a locally Minkowski space M^2 and the fundamental function $L(x,y;p,q)$ is written in an adapted coordinate system (x,y) of M^2 as $L = e^{f(x,y)} N(p,q)$, $f = a_1 x + a_2 y + a$ with constant a's, where N is the fundamental function of M^2 in (x,y).*

Remark 1. In the Riemannian case we have the notion of "isothermal coordinates" in the two-dimensional case. In such a coordinate system the fundamental function $L(x,y;p,q)$ can be written as

$$L = e^{f(x,y)} \sqrt{p^2 + q^2}$$

as above. (Therefore, the equation (1.9) is not a condition for the Riemannian case.)

Consequently our theorem shows

Corollary. *Let R^2 be a two-dimensional Riemannian space with the fundamental form $ds^2 = e^{2f(x,y)}(dx^2 + dy^2)$ in an isothermal coordinate system (x, y). All the Christofell symbols of R^2 are constant in (x, y), if and only if $f(x, y) = a_1 x + a_2 y + a$ with constant a's.*

It is clear that the condition "y-Berwald" is equivalent to "constant-Berwald" for Riemannian metrics.

Remark 2. It seems that the notion of a y-Berwald space was first introduced in 1991 by the first author [1]. Contrasting with this notion, a Finsler space with $G^i_{jk} = G^i_{jk}(x)$ is called a *Berwald space* and we have an extensive literature on these spaces. As is well-known [2], a Berwald space can be characterized in terms of the Cartan connection $C\Gamma$: A Finsler space is a Berwald space, if and only if the connection coefficients Γ^{*i}_{jk} of $C\Gamma$ are functions of x^i alone. Thus we have an interesting question from the standpoint of geometry: How about Γ^{*i}_{jk} of y-Berwald spaces?

It follows immediately from the well-known equation $y^j\Gamma^{*i}_{jk} = G^i_k$ [2] that $\Gamma^{*i}_{jk} = \Gamma^{*i}_{jk}(y)$ implies $G^i_{jk} = G^i_{jk}(y)$; the space is y-Berwald.

In the two-dimensional case the inverse is also true. In fact, as already shown, $L(x, y; p, q)$ of y-Berwald space F^2 is written as $L = e^{f(x,y)}N(p, q)$ in an adapted coordinate system (x, y), where $N(p, q)$ is the fundamental function of a locally Minkowski space M^2. Since F^2 is conformal to M^2, F^2 has the common tensor $C^i_{jk}(p, q)$ with M^2 [2], (3.4.1.3′). Then the components of the tensor

$$\begin{aligned} C^i_{jk;0} = {} & \{\partial C^i_{jk}/\partial x^h - (\partial C^i_{jk}/\partial y^r)G^r_h\}y^h \\ & + C^r_{jk}G^i_r - C^i_{rk}G^r_j - C^i_{jr}G^r_k \end{aligned}$$

are also functions of (p, q) alone. Hence the equation $\Gamma^{*i}_{jk} = G^i_{jk} - C^i_{jk;0}$ [2], (2.5.2.7) shows $\Gamma^{*i}_{jk} = \Gamma^{*i}_{jk}(y)$. Therefore we have

Proposition. (1) *If a Finsler space F^n is covered by coordinate neighborhoods in each of which the coefficients Γ^{*i}_{jk} of the Cartan connection are functions of y^i alone, then F^n is a y-Berwald space.* (2) *A two-dimensional Finsler space F^2 is a y-Berwald space, if and only if there exists a covering of coordinate neighborhoods in each of which Γ^{*i}_{jk} are functions of y^i alone.*

2. Examples

As shown in [7], (2.7) or [8], just before (1.6), the equation of the geodesics is written, in a two-dimensional Finsler space with coordinates

(x, y), in the form

$$y'' = 2y'G^1(x, y; 1, y') - 2G^2(x, y; 1, y'), \quad y' = dy/dx.$$

Hence, in a y-Berwald space this is of the form $y'' = f(y')$, and it may well be that a y-Berwald metric will be found by the *metrization* of such a differential equation.

In expectation of this hope we shall consider the following examples.

Example 1. We first deal with the differential equation

$$y'' + y' + 1 = 0. \tag{2.1}$$

Let us find the two-dimensional Finsler metric $L(x, y; p, q)$ whose geodesics are given by (2.1).

The solution of (2.1) is written as

$$y := \phi(x) = ae^{-x} - x + b, \tag{2.2}$$

with arbitrary constants (a, b). This is the finite equation of the family of geodesics.

Following the method shown in [7] or [8], we find successively functions $\alpha(x, y, z)$, $\beta(x, y, z)$, $u(x, y, z)$, $U(x; a, b)$, $V(x, y, z)$ and $B(x, y, z)$:

$$z := y' = -ae^{-x} - 1, \quad \alpha := a = -e^x(z+1), \quad \beta := b = x + y + z + 1,$$
$$u := y'' = -(z+1), \qquad U := \exp \int u_z(x, \phi, \phi_x)dx = e^{-x},$$
$$V := U(x; \alpha, \beta) = e^{-x}.$$

Consequently we get

$$B(x, y, z) := H(\alpha, \beta)/V(x, y, z) = e^x H(\alpha, \beta), \tag{2.3}$$

where H is an arbitrary function. Then the associated fundamental function $A(x, y, z) := L(x, y; 1, z)$ is written in the form

$$\begin{aligned} A &= A^*(x, y, z) + C(x, y) + D(x, y)z, \\ A^* &= \iint B(x, y, z)(dz)^2, \end{aligned} \tag{2.4}$$

where C and D are arbitrary functions but should be chosen to satisfy

$$C_y - D_x = A^*_{zz}u + A^*_{yz}z + A^*_{xz} - A^*_y. \tag{2.5}$$

Finally we obtain the fundamental function

$$L(x, y; p, q) = pA(x, y, q/p). \tag{2.6}$$

Now, let us take $H(\alpha,\beta)=1$ for simplicity. Then we have

$$A^* = e^x z^2/2, \quad C_y - D_x = -e^x.$$

Choosing $C=0$ and $D=e^x$, we finally obtain $A=e^x(z^2/2+z)$ and the metric

$$L(x,y;p,q) = e^x(q^2/2p+q). \tag{2.7}$$

This is certainly a y-Berwald metric according to the above Theorem; in fact G^i are given by (1.11) as follows:

$$\begin{aligned} 2G^1 &= p^2\{1-2(p+q)^2/q(2p+q)\},\\ 2G^2 &= pq\{1-(p+q)/(2p+q)\}. \end{aligned} \tag{2.8}$$

Example 2. We shall be concerned with the differential equation

$$y'' + (y')^2 + y' = 0, \tag{2.9}$$

of the Liouville type. The solution is written as

$$y = \log\ |ae^{-x}+b|, \tag{2.10}$$

with arbitrary constants (a,b).

Similarly as in Example 1, we have

$$\begin{aligned} &z=-a/(a+be^x), &&\alpha=-ze^{x+y}, &&\beta=(z+1)e^y,\\ &u=-(z^2+z), &&U=e^{-x}(ae^{-x}+b)^{-2}, &&V=e^{-(x+2y)}. \end{aligned}$$

Thus we get

$$B = e^{x+2y}H(\alpha,\beta). \tag{2.11}$$

We are especially interested in $H(\alpha,\beta)=\alpha^n$.

(1°) $n \neq -1,-2$. Then double integration leads to

$$A^* = (-1)^n z^{n+2}\ \exp\{(n+1)x+(n+2)y\}/(n+1)(n+2), \quad C_y - D_x = 0.$$

Taking $C=D=0$, we get $A=A^*$. Thus, within the constant factor $(-1)^n/(n+1)(n+2)$ we obtain

$$L(x,y;p,q) = q^{n+2}p^{-n-1}\exp\{(n+1)x+(n+2)y\}. \tag{2.12}$$

Though this is obviously a y-Berwald metric, it is only a *locally Minkowski metric,* because in $(\bar{x},\bar{y})=(e^{-x},e^y)$ we have the metric of the form $L = (-1)^{n+1}(\bar{q})^{n+2}(\bar{p})^{-n-1}$, and the geodesics (2.10) reduce to straight lines $\bar{y}=a\bar{x}+b$.

(2°) $n=-1$. Then we have

$$A^* = -ze^y(\log|z|-1), \quad C_y - D_x = e^y.$$

Taking $C=e^y$ and $D=0$, we obtain

$$L(x,y;p,q) = e^y(p+q-q\ \log|q/p|). \tag{2.13}$$

This is, of course, a y-Berwald metric: G^i are written as

$$\begin{aligned} 2G^1N &= pq(p+q+p\ \log|q/p|),\\ 2G^2N &= q^2(3p+2q+p^2/q-q\ \log|q/p|),\\ N &= p+q-q\ \log|q/p|. \end{aligned} \tag{2.14}$$

(3°) $n=-2$. Then we have

$$A^* = -e^{-x}\log|z|, \quad C_y - D_x = e^{-x}.$$

Choosing $C=0$ and $D=-e^{-x}$, we obtain

$$L(x,y;p,q) = e^{-x}(p\ \log|q/p|+q). \tag{2.15}$$

This y-Berwald metric has G^i of the form

$$\begin{aligned} 2G^1 &= -p^2-(p+q)^2p/N,\\ 2G^2 &= -pq+(p+q)pq(\log|q/p|-1)/N,\\ N &= p\ \log|q/p|+q. \end{aligned} \tag{2.16}$$

Thus we obtain two y-Berwald metrics (2.13) and (2.15). They are both projectively flat, because they have the common geodesics with the locally Minkowski metric (2.12).

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