A note on the idealizer of a subring.

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A familiar notion in group theory is the normalizer of a subgroup H of G defined as the greatest subgroup of G in which H is a normal subgroup. The corresponding concept in ring theory: the "idealizer" of a subring has been recently introduced in a special case by Prof. L. KALMÁR[1]). The purpose of the present note is to give an application of this new concept and to show that it is a useful notion in algebra.

By the *idealizer* of a subring T of a commutative ring R we shall mean the greatest subring S of R in which T is an ideal. This definition of idealizer may be reformulated in a more direct way if we introduce the ring-residual $T:\sigma$ of a subring T by an arbitrary set σ of elements of R: let $T:\sigma$ be the set of all $x \in R$ satisfying $x\sigma \subseteqq T$. (The ring-residual is readily seen to be a subgroup of R^+.) Now the idealizer may alternatively be defined as the ring-residual $T:T$. It is immediate that the ring-residual and hence the idealizer always exists.

Let R be a commutative ring and Q the quotient-ring[2]) of R. Any ideal A of R may be considered as a subring of Q and so we may form the idealizer of A in Q. This is a subring $I(A)$ of Q containing R.

Theorem 1. *A ring R is integrally closed*[3]) *in its quotient-ring if and only if R is the idealizer of each finite regular ideal*[4]) *A of R.*

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[1]) L. KALMÁR, Über die Cantorsche Theorie der reellen Zahlen. *Publicationes Mathematicae,* 1 (1950), pp. 150–159.

[2]) The *quotient-ring* Q of R contains, besides the elements of R, all fractions a/b with a, b in R such that b is regular (no divisor of zero). [The usual definition of quotient-rings (e. g. B. L. VAN DER WAERDEN, Moderne Algebra, vol. I (1937), p. 45, or W. KRULL, Idealtheorie (1935), p. 19) leaves out of consideration the degenerated case of rings without regular elements; in this case the quotient-ring is empty according to the usual definition, while by completion with the term "besides the elements of R" it coincides with the original ring R.]

[3]) By definition, R is *integrally closed* in its quotient-ring Q, if an algebraic equation $x^n + c_1 x^{n-1} + \ldots + c_n = 0$ with $x \in Q$, $c_1, \ldots, c_n \in R^*$ implies $x \in R$. Here R^* denotes a least overring of R with a unit element (if $1 \in R$, $R^* = R$), that is, R^* consists of all pairs (r, n) ($r \in R$,

In the proof we may restrict ourselves to the case if R contains at least one regular element, since otherwise the statement is trivial.

Let R be integrally closed, hence having a unit element, and let $A=(a_1,\ldots,a_n)$ be a finite regular ideal of R. If x belongs to $A:A=I(A)$, then each element of xA is of the form $r_1a_1+\ldots+r_na_n$ with $r_i\in R^*=R$. Therefore $xa_i=r_{i1}a_1+\ldots+r_{in}a_n$ with $r_{ik}\in R$, and hence the determinant $\Delta=|r_{ik}-\varepsilon_{ik}x|$ (where ε_{ik} is equal to 0 or 1 according as $i\neq k$ or $i=k$) is an annihilator of A. Consequently, by the regularity of A, we get $\Delta=0$. Hence we have an algebraic equation for x with coefficients in R and leading term x^n; thus x is integral over R. By integral closure, $x\in R$, that is, $I(A)=R$.

Conversely, if the idealizer of each finite regular ideal is R, then for each $x\in Q$ satisfying an algebraic equation $x^n+c_1x^{n-1}+\ldots+c_n=0$ $(c_i\in R^*)$, we form the (fractional[5]) ideal $A=(1,x,\ldots,x^{n-1})$ which is finite and regular. Now $xA=(x,x^2,\ldots,x^n)=(x,x^2,\ldots,-c_1x^{n-1}-\ldots-c_n)\subseteq A$, and since by hypothesis $A:A=R$, we find $x\in R$, i. e. R is integrally closed, in fact. The proof is completed.

We define a prime ideal P of R to be *complete* if it is divisorless and at the same time $I(P)=R$; we may then prove

Theorem 2. *Let A be an ideal of an integral domain R with maximal condition. If all prime overideals of A are complete, then A may be represented uniquely as the product of prime ideals.*

For the proof we may proceed on the lines of B. L. VAN DER WAERDEN's proof in his cited book[3]) § 102. It is only to be noted that for a complete prime ideal P always $P\cdot P^{-1}=R$ holds. Indeed, otherwise $P\cdot P^{-1}=P$, and this would imply $P^{-1}\subseteq P:P=I(P)=R$, i. e., $P^{-1}=R$ which is absurd, considering that P^{-1} for each prime ideal P necessarily contains elements not in R[6]).

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n a rational integer) with the composition rules $(r_1,n_1)+(r_2,n_2)=(r_1+r_2,n_1+n_2)$ and $(r_1,n_1)(r_2,n_2)=(r_1r_2+n_2r_1+n_1r_2,n_1n_2)$. The subrings $(r,0)$ and $(0,n)$ can be identified with R and the ring of all rational integers, respectively, so that $(r,n)=r+n$ is actually a sum in R^*. Our definition of integral closure coincides with VAN DER WAERDEN's terminology (Moderne Algebra, vol. II (1940), pp. 76–77) for rings with maximal condition. It is immediate that, whenever R contains at least one regular element, integral closure implies the existence of a unit element as the root of the equation $x^2-x=0$.

[4]) An ideal A is *finite*, if it has a finite base and is *regular* if it contains at least one regular element.

[5]) A fractional ideal may be made into an integral ideal by multiplication by a regular element. Hence if $I(A)=R$ holds for all finite integral regular ideals A, then the same must hold for all fractional ideals of the same type.

[6]) See loc. cit.[3]), § 102, Hilfssatz 3.