# MATHEMATICAL NOTES.

By ALFRÉD RÉNYI in Debrecen.

## I. On the summability of Cauchy-Fourier series.

Let us consider a real measurable odd function $f(t) = -f(-t)$ defined in $(-\pi, \pi)$, L-integrable in every interral $(\varepsilon, \pi)$, $\varepsilon > 0$ and for which the integral

$$\int_0^\pi t f(t)\, dt \tag{1}$$

exists; we do not suppose however $f(t)$ to be integrable in $(0, \pi)$. The existence of (1) implies the existence of

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(t) \sin nt\, dt \tag{2}$$

for $n = 1, 2, \ldots$. The formulae

$$\int_{-\pi}^{+\pi} f(t) \cos nt dt = 0, \qquad n = 0, 1, 2, \ldots \tag{3}$$

are also valid, by virtue of $f(t) = -f(-t)$ when considered as *Cauchy principal values*. If these conditions are satisfied, we shall call the series

$$\sum_{n=1}^{\infty} b_n \sin nt \tag{4}$$

*the Cauchy- Fourier (sine-) series of* $f(t)$. CAUCHY-FOURIER series as defined above has been considered by TITCHMARSH,[1]) who called them „principal-value Fourier series". In this note we shall show that the theorem of FEJÉR-LEBESGUE[2]) remains valid also for *Cauchy-Fourier* series, i. e. that (4) is summable by *Fejér* means to $f(t)$ almost everywhere, especially in every con-

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[1]) E. C. TITCHMARSCH, Principal value Fourier series, *Proc. London Math. Soc.* (2) **23**, (1925), 41—43.

[2]) L. FEJÉR, Sur les fonctions bornées et intégrables, *Comptes Rendus Acad. Sci. Paris*, 1900, 984–987; H. LEBESGUE, *Math. Ann.* **61**, (1905) 251—280. .

tinuity point of $f(t)$. This has been stated (without proof) by TITCHMARSH (l. c. 1)). We begin by remarking, that

(5) $$b_{n+1}-b_{n-1}=o(1).$$

As a matter of fact, we have

(6) $$b_{n+1}-b_{n-1}=\frac{4}{\pi}\int_0^\pi f(t)\sin t\cos nt\,dt$$

and (5) follows by RIEMANN's well known lemma, as $f(t)\sin t$ is L-integrable. It follows from (5) immediately that

(6) $$b_n=o(n)$$

Now let us put $g(t)=f(t)\sin t$; $g(t)$ is evidently even and integrable in $(-\pi,\pi)$. Let $\sigma_n(f;t)$ denote the $n$-th arithmetic mean of the series (4) and $\sigma_n(g;t)$ that of the FOURIER series of $g(t)$. A simple calculacion gives

(7) $$\sigma_n(g;t)=\sin t\ \sigma_n(f;t)+R_n(t)+r_n(t)$$

where

(8) $$R_n(t)=\frac{\operatorname{ctg} t}{2(n+1)}\left[\sum_{k=2}^{n-2}(b_{k-1}-b_{k+1})\sin kt+b_{n-2}\sin(n-1)t+b_{n-1}\sin nt\right]$$

and

(9) $$r_n(t)=\frac{b_{n+1}\cos nt+b_n\cos(n+)t-b_2\cos t-2b_1}{2(n+1)}$$

By virtue of (5) and (6), $R_n(t)$ and $r_n(t)$, defined by (8) and (9) tend to 0 for $n\to\infty$, uniformly in every closed interval not containing any of the points $-\pi,0,\pi$. Thus it follows that if $t\not\equiv 0 \bmod \pi$ and $\sigma_n(g;t)$ converges to $g(t)$ then $\sigma_n(f;t)$ converges also to $\frac{g(t)}{\sin t}=f(t)$; as $\sigma_n(g;t)$ converges almost everywhere and especially in every continuity point of $g(t)$, the same holds for $f(t)$; similarly it follows that if $f(t)$ is continuous in a closed interval not containing any of the points $-\pi,0,+\pi$ the series (4) is uniformly summable in this interval to $f(t)$.

The above results are contained in the (unpublished) doctor's thesis of the author (Szeged, 1945) who is thankful to R. SALEM, whose valuable remarks helped him to shorten the above proof.

It is to be mentioned that the convergence of (4) is out of question, because the coefficients of a CAUCHY-FOURIER series do not tend to zero; as a matter of fact (6) can not be improved without imposing further conditions on $f(t)$.

An other possible approach to the investigation of CAUCHY-FOURIER series can be obtained by remarking that the series

(10) $$A_0-\sum_{n=t}^{\infty}\frac{b_n}{n}\cos nt$$

where

$$A_0 = \frac{1}{\pi}\int_0^{\pi} t f(t)\,dt \tag{11}$$

is the *Fourier* series of the integrable even function $F(t)$ defined for $t>0$ by

$$F(t) = \int_t^{\pi} f(u)\,du. \tag{12}$$

These remarks follow from the fact that $F(t)$ defined by (12) is L-integrable and[3])

$$\int_0^{\pi} F(t)\,dt = \int_0^{\pi} t f(t)\,dt \tag{13}$$

further from the relation

$$\lim_{t\to 0} t F(t) = 0$$

which is a simple consequence of the definition of $F(t)$. Thus the CAUCHY-FOURIER series (4) can be obtained by term-by-term derivation of an ordinary FOURIER series. In this way the summability of (4) can be deduced from general theorems of W. H. YOUNG[4]) but we preferred to give a straightforward proof. As examples of CAUCHY-FOURIER series we mention

$$\frac{\cos (D-2r)\frac{t}{2}}{2\sin D\frac{t}{2}} \sim \sin rt + \sin (D+r)t + \sin (2D+r)t + \ldots \tag{15}$$

where $D$ and $r$ are arbitrary positive integers.

*(Recived March 8, 1950.)*

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[3]) HARDY, LITTLEWOOD and PÓLYA, Inequalities, (Cambridge 1934), p. 169, example 242.

[4]) W. H. YOUNG, On the convergence of the derived series of Fourier series, *Proc. London Math. Soc.* (2) **17.** (1918), 195—236. -- I am thankful to prof F. RIESZ for calling my attention to the papers of W. H. YOUNG. In a special case the summability of the derived series has been proved by L. FEJÉR, Math. Ann. **58.** (1904) 51—69.