On two problems concerning the theory of binary relations.

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Let us consider the closed interval [0, 1] Let us suppose that to each pont x of this interval there corresponds a set S_x — called the picture of x — the points of which are outside of the interval $K(x)$ which is symmetrical about x; the length of $K(x)$ shall be denoted by $f(x)$; we shall suppose only that $f(x) > 0$, for every x. Let us call two points of the interval [0, 1] *independent*, if neither of them belongs to the picture of the other.

Problem 1. Does there exist in [0, 1] a set of positive measure, all pairs of points of which are independent?

Problem 2. Does there exist in [0, 1] a set, having the power of the continuum, all pairs of points of which are independent?

Theorem 1. *If $f(x)$ is a measurable function, there exists in* [0, 1] *a set of positive measure, the pairs of points of which are independent.*

Proof. The sets

$$E_i = E\left[\frac{1}{i} < f(x) \leq \frac{1}{i-1}\right] \qquad (i = 2, 3, \ldots)$$

cannot all be of measure zero, because in this case the interval [0, 1] would be the sum of countably many sets of measure zero. Hence there exists a value $i = i_1$ for which E_{i_1} is of positive measure. Let us divide the interval [0, 1] into n subintervals, by means of the points $x_0 = 0 < x_1 < x_2 < \ldots < x_n = 1$, in such a manner that the length of each subinterval (x_k, x_{k+1}) should be less than $\frac{1}{2i_1}$. Let us now denote the common part of the set E_{i_1} with the subinterval $[x_k, x_{k+1}]$ by E_{i_1k} $(k = 0, 1, 2, \ldots, n-1)$. Among the sets E_{i_1k} there must be at least one — $E_{i_1k_1}$ say — having positive measure, because the sum of the sets E_{i_1k} is of positive measure. Any two points of $E_{i_1k_1}$ are independent, because if x belongs to $E_{i_1k_1}$ and y to S_x we have $|y-x| > \frac{f(x)}{2} \geq \frac{1}{2i_1}$ and thus y is outside $E_{i_1k_1}$.

Theorem 2. *The answer to problem 2 is always in the affirmative.*[1])

Proof. Let us define the sets E_i as above $(i = 2, 3, \ldots)$. If all sets E_i were countable, the interval $[0, 1]$ would be the sum of countably many countable sets. Thus there exists a value $i_1 = i$ for which E_{i_1} has the power of the continuum. Let us now define the points x_k and the sets $E_{i_1 k}$ as above $(k = 0, 1, 2, \ldots, n-1)$. At least one of the sets $E_{i_1 k}$ — say $E_{i_1 k_1}$ — will have the power of the continuum. It follows exactly as above that any two points of $E_{i_1 k_1}$ are independent.

So far we supposed, that the interval $K(x)$ is symmetrical about x. It is easy to see that our results are true and the proofs essentially unchanged. if we suppose only that x is an interior point of the interval $K(x)$, i. e. $K(x)$ is the interval $[x - f_1(x), x + f_2(x)]$ where $f_1(x)$ and $f_2(x)$ are positive functions; as regards Theorem 1 it must be supposed that both $f_1(x)$ and $f_2(x)$ are mesurable. As a matter of fact, if $f(x) = \min(f_1(x), f_2(x))$, it follows that S_x is outside of the symmetrical interval of length $2f(x)$ about x, and the problem is reduced to that considered above.

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[1]) D. Lázár, On a problem in the theory of aggregates. *Compositio Math.* 3 (1936), 304. In this paper D. Lázár has proved a similar theorem in the case where each S_x has only a finite number of elements. His method could be applied also to our case, but we prefer to give a different proof.