On semigroups admitting relative inverses and having minimal ideals.

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1. A. SUSCHKEWITSCH[1]) was the first to consider the structure of semigroups. He has proved that every finite semigroup S has a uniquely determined subsemigroup K, called its *kernel* (Kerngruppe), which decomposes into the class-sum of disjoint isomorphic groups $C_{\varkappa\lambda}$ such that[2])

$$K=\sum_{\varkappa=1}^{r} A_{\varkappa}=\sum_{\lambda=1}^{s} B_{\lambda}=\sum_{\varkappa=1}^{r}\sum_{\lambda=1}^{s} C_{\varkappa\lambda},\quad A_{\varkappa}=\sum_{\lambda=1}^{s} C_{\varkappa\lambda},\quad B_{\lambda}=\sum_{\varkappa=1}^{r} C_{\varkappa\lambda},$$

where $A_{\varkappa}$ (B_{λ}) are isomorphic subsemigroups with the right-(left-)cancellation law[3]). Later A. H. CLIFFORD[4]) has considered infinite semigroups admitting relative inverses and prcved that such a semigroup is the class-sum of mutually disjoint groups. However, about the structure of the kernel his results[5]) do not give such an exhaustive information as the structure theorem of SUSCHKEWITSCH cited above.

In the present paper we show how to extend SUSCHKEWITSCH' results to infinite semigroups admitting relative inverses which contain minimal right-ideals. A simple generalization of the kernel to which our method naturally leads is set out as Theorem 7 in section 5.

2. Let S be a semigroup, i. e., a system closed under an associative operation and let S admit relative inverses in the sense of A. H. CLIFFORD

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[1]) A. SUSCHKEWITSCH, Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit. *Math. Ann.*, **99** (1928), 30–50.

[2]) Σ denotes class-sum.

[3]) The kernel is, in the terminology of D. REES [On semi-groups. *Proc. Cambr. Phil. Soc.*, **36** (1940), 387—400], a completely simple semigroup without zero, i. e, a semigroup K such that, for given a, b in K, the equation $xay=b$ is always solvable for x and y in K.

[4]) A. H. CLIFFORD, Semigroups admitting relative inverses. *Ann. of Math.* (II. s.), **42** (1941), 1037—1049. For the definition of relative inverse see below 2.

[5]) See also A. H. CLIFFORD, Semigroups containing minimal ideals. *Amer. J. Math.*, **70** (1948), 521–526.

(loc. cit.[4])): to each a in S there exists an element e and a relative inverse a' such that $ea = ae = a$, $aa' = a'a = e$. It is easily proved that this e is idempotent and is uniquely determined by a, further, if a' is a relative inverse of a, then $a^{-1} = ea'e$ is also one satisfying $ea^{-1} = a^{-1}e = a^{-1}$ and a^{-1} is unique. When we speak about the relative inverse of a, we shall always understand this a^{-1}. Clearly, a is the relative inverse of a^{-1}.

A subset A of S is called a right-(left-)ideal, if[6]) $AS \subseteq A$ $(SA \subseteq A)$; A is a two-sided ideal if $SAS \subseteq A$. The set aS (Sa, SaS) is said to be the principal right-(left-, two-sided) ideal generated by a. Since each element has its own two-sided unit, it follows that every principal ideal contains its generating element.

We define a subdivision of S into disjoint classes by the specification that a and b belong to the same class C_σ if, and only if, they generate the same principal right-ideal, i. e., $aS = bS$. In order to discover the structure of C_σ we prove a series of lemmas.

Lemma 1. *C_σ is a subsemigroup of S.*

It is to be proved that C_σ contains together with a, b also ab. Suppose a, b to belong to C_σ, that is, $aS = bS$. We have evidently $abS \subseteq aS$ as well as $abS = aaS \supseteq aaa^{-1}S = aeS = aS$. These inclusions imply $abS = aS$, i. e., $ab \in C_\sigma$ as stated.

Lemma 2. *C_σ admits relative inverses.*

For, if e is the idempotent belonging to a and a^{-1} is the relative inverse of a in S, then $aS \supseteq aa^{-1}S = eS \supseteq ea^{-1}S = a^{-1}S$ and the dual inclusions $a^{-1}S \supseteq eS \supseteq aS$ imply that both e and a^{-1} generate the same right-ideal as a, hence $e, a^{-1} \in C_\sigma$. This lemma admits of writing each principal right-ideal with an idempotent generating element.

Lemma 3. *C_σ is a left-right system in the sense defined by* H. B. Mann[7]).

It is to be shown that C_σ contains a universal left-unit and each element has a right-inverse with respect to this left-unit. $a, b \in C_\sigma$, that is, $aS = bS$ implies the existence of an $x \in S$ such that $ax = b$ Let e be the idempotent belonging to a; then $eb = eax = ax = b$, that is to say, e and hence every idempotent of C_σ is a universal left-unit for C_σ[8]). To complete the proof, we show that an equation $ax = u$ ($a \in C_\sigma$, u a left-unit in C_σ) is solvable for $x \in C_\sigma$. Put $x = a^{-1}u$, then $x \in C_\sigma$ and $aa^{-1}u = eu = u$, e being a left-unit for C_σ.

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[6]) $\subseteq$ denotes inclusion, $\subset$ denotes proper inclusion.

[7]) H. B. Mann, On certain systems which are almost groups. *Bull. Amer. Math. Soc.*, **50** (1944), 879—881. A left-right system, or, briefly a (l, r)-system, is closed under a binary associative operation and contains a left-unit and right-inverses.

[8]) More generally, every idempotent e of C_σ is a universal left-unit for the whole right-ideal $aS = bS$.

Lemma 4. *In C_σ the left-cancellation law holds.*

For, multiply the equation $ax=ay$ $(a, x, y \in C_\sigma)$ by a^{-1} on the left to get $ex=ey$, whence $x=y$, since the idempotent e belonging to a is a left-unit for all the elements of C_σ.

From Lemmas 1—4 we conclude:

Theorem 1. *A semigroup admitting relative inverses decomposes into the class-sum of mutually disjoint (l, r)-systems C_σ with the left-cancellation law, which are themselves semigroups admitting relative inverses.*

Let R be any right-ideal of S. If R contains a single element a of C_σ, then R contains the principal right-ideal generated by a, hence R wholly contains C_σ. Thus we are led to

Theorem 2. *Any right-ideal R of S splits into the class-sum of mutually disjoint (l, r)-systems C_σ with the left-cancellation law.*

Of course, the same holds for left-ideals L using (r, l)-system D_τ with the right-cancellation law instead of (l, r)-systems.

3. A (l, r)-system C_σ and a (r, l)-system D_τ define a cross-cut $E_{\sigma\tau}=C_\sigma \cap D_\tau$. If this is not empty, then it contains with each a also a^{-1} and e. e being a left-unit for C_σ and at the same time a right-unit for D_τ, e is the identity element of $E_{\sigma\tau}$. $E_{\sigma\tau}$ is plainly a group. Since a group can not have other idempotents than its identity, it follows that $E_{\sigma\tau}$ contains no element belonging to an idempotent other than the identity e of $E_{\sigma\tau}$. On the other hand, both C_σ and D_τ contain together with e also every element belonging to the idempotent e. Hence every $E_{\sigma\tau}$ which is not vacuous, consists of all elements belonging to the same idempotent e, consequently, it coincides with one of the groups S_e defined by CLIFFORD (loc. cit. [4])). We thus conclude that each (l, r)-system C_σ $[(r, l)$ system $D_\tau]$ consists of certain of the groups S_e.

Now the problem arises as to which groups S_e belong to the same C_σ. This may be settled by the idempotents: it is obvious that C_σ contains together with e also all idempotents e' for which $ee'=e'$ and $e'e=e$, but only these idempotents[9]). Hence a (l, r)-system C_σ consists of a set of the groups S_e such that the idempotents e are exactly those which are left-units for each other.

Let e be a left-unit in C_σ and consider the set $C_\sigma e$, that is, the set of all elements of C_σ whose left- and right-unit is e. This set evidently coincides with the group S_e of C_σ. The mapping $x \to xe$ induces a homomorphism of C_σ onto S_e, for $a \to ae$, $b \to be$ imply $ab \to abe = a(eb)e = (ae)(be)$. Moreover: the groups $S_e, S_{e'}$, of which C_σ consists are isomorphic under the correspondence $ae \leftrightarrow ae'$ $(a \in C_\sigma)$. In fact, this is one-to-one: ae is

[9]) If e', e'' are two such idempotents, then they are left-units for each other, for $e'e''=e'(ee'')=(e'e)e''=ee''=e''$ and similarly $e''e'=e'$.

mapped upon $aee' = ae'$ and vice versa, further, it is a homomorphism: $(ae)(be) = abe \leftrightarrow abe' = (ae')(be')$, since e as well as e' are universal left-units for C_σ. Hence:

Theorem 3. *Each (l, r)-system C_σ consists of isomorphic groups S_e such that the idempotents e are left-units for each other*[10]).

4. Assume that there is a minimal right-ideal R in S. By the minimality of R, any element of R generates R, hence R consists of one C_σ, say $R = C^*$. Let e be an idempotent belonging to R (thus $R = eS$) and consider the principal left-ideal Se. This is also minimal. To verify this we shall show that any principal left subideal Sf of Se equals Se (f an idempotent). Sf has a non-vacuous intersection with R, for ef belongs to both of them; therefore there is a group S_g with the identity g in $R \cap Sf$. This implies $Sg \subseteq Sf \subseteq Se$, and hence e is a right-unit for g, $ge = g$. On the other hand, e and g being idempotents in $R = C^*$, g is a left-unit for e, thus $ge = e$. Hence we get $g = e$; consequently, $Sf = Se$ and Se is a minimal left-ideal, indeed. We have thus proved:

Theorem 4. *An element generates a minimal left-ideal if and only if it generates a minimal right-ideal.*

Let $\{e_\lambda\}$ be the set of idempotents contained in the minimal right-ideal R of S, and consider the left-ideals $L_\lambda = Se_\lambda$ generated by e_λ. Theorem 4 implies that all the L_λ are minimal. Moreover, there is no minimal left-ideal beside these L_λ, for each left-ideal must have a non-void intersection with the right-ideal R and hence also an idempotent in common with R. The same inference shows that the right-ideals generated by the idempotents of one of L_λ exhaust all minimal right-ideals $R_\varkappa$ of S. Hence it follows at once that the class-sum K of all minimal right-ideals $R_\varkappa$ is the same as that of all minimal left-ideals L_λ. It is also readily seen that K is a two-sided ideal: the unique minimal two-sided ideal of S contained in every two-sided ideal of S. K is said to be the *kernel* of S. The kernel has the same structure as in the finite case discovered by A. SUSCHKEWITSCH:

Theorem 5. *If S has a minimal right-ideal, then the kernel K of S exists. K has the following structure:*

$$K = \sum_\varkappa R_\varkappa = \sum_\lambda L_\lambda = \sum_\varkappa \sum_\lambda S_{\varkappa\lambda}, \quad R_\varkappa = \sum_\lambda S_{\varkappa\lambda}, \quad L_\lambda = \sum_\varkappa S_{\varkappa\lambda},$$

where the $R_\varkappa$ (L_λ) are isomorphic (l, r)-systems with the left-cancellation law [(r, l)-systems with the right-cancellation law] and $S_{\varkappa\lambda} = R_\varkappa \cap L_\lambda$ are isomorphic groups.

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[10]) This corresponds to MANN's result (loc cit. [7])) according to which each (l, r)-system is the direct product of a group and an idempotent (l, r)-system.

We have to verify the isomorphisms. The fact that the groups belonging to minimal one-sided ideals are all isomorphic is an immediate consequence of Theorem 3. This isomorphism of the groups $S_{\varkappa\lambda}$ may be extended to an isomorphism of the right-(left-)ideals $R_\varkappa$ (L_λ) such that the groups $S_e \in R$ and $S_{e'} \in R'$ correspond to each other if, and only if, they belong to the same left-ideal $Se = Se'$; further, let $xe' \in R'$ correspond to $x = xe \in R$. This correspondence is then actually an isomorphism between R and R', for if $xe \leftrightarrow xe'$ $(xe \in R,\ xe' \in R')$ and $yf \leftrightarrow yf'$ $(f^2 = f,\ f'^2 = f',\ yf \in R,\ yf' \in R')$, then $xe \cdot yf = xyf \in S_f \subseteq R$ and $xe' \cdot yf' = xyf' \in S_{f'} \subseteq R'$ are corresponding elements. This completes the proof of the isomorphisms.

5. The above results on minimal right-ideals may be generalized to a certain class of ideals. By the *rank* of a principal right-ideal aS we shall understand 1 if aS is minimal, and $n+1$ if $aS \supset bS$ implies that the rank of bS is at most n, but not at most $n-1$ $(n > 1)$. It is immediate that by this definition every principal right-ideal of rank n contains a principal right-ideal for every rank $\leqq n-1$, but neither another one of the same rank nor one of a rank greater than n. The same definition applies to left-ideals.

Theorem 6. *The principal right-ideal eS is of rank n if and only if the principal left-ideal Se is of rank n.*

Proof by induction on n. For $n = 1$ this is just the statement of Theorem 4. Let the rank n of eS be greater than 1 and suppose the assertion true for ideals of a rank $< n$ By the induction hypothesis the principal left-ideal Se can not be of a rank less than n, consequently, there is a principal left-ideal Sf (f an idempotent) of rank n contained in Se. Clearly, $fe = f$ and hence the left-ideal Sef contains $fef = ff = f$, so that $Sef = Sf$. On the other hand $efS \subseteq eS$ and, if efS had a rank $\leqq n-1$, then, by induction, $Sef = Sf$ would have the same rank against the assumption on Sf; hence efS is of rank n. This implies $efS = eS$, ef is a left-unit for e, that is, $efe = e$. But we have $efe = e(fe) = ef$, consequently, $ef = e$ and hence $Sf = Sef = Se$, as we wished to prove.

Now assuming the existence of a principal right- or left-ideal of rank n, by the method of section 4 one gets kernels of rank n. The above discussions imply

Theorem 7. *Each kernel of higher rank has the same structure as* SUSCHKEWITSCH' *kernel.*

The only, but basic difference between the kernel of rank 1 and the kernels of higher rank lies in the fact that several kernels of the same rank > 1 may exist; moreover, in the simplest finite cases the kernels of higher rank need not be unique!

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