On quasi-linear functional operations.

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An operation $L[f]$ on the linear real functional space F is linear if

(1) $L[af+g]=aL[f]+L[g]$ (a is an arbitrary real constant, $f,g\in F$)

and $L[f]$ is continuous. — F. RIESZ, M. FRÉCHET and M. H. STEINHAUS[1]) gave the general form of linear functional operations in different functional spaces F (the space of continuous functions; the L^p space) as a Stieltjes integral, resp. a common Lebesgue integral.

We call an operation $U[f]$ quasi-linear[2]) if, besides being continuous, it satisfies the condition

(2) $U[af+g]=U[af^*+g]$ whenever $U[f]=U[f^*]$

(This implies of course also $U[af+g]=U[af^*+g^*]$ and even $U[af+bg]= =U[af^*+bg^*]$ whenever $U[f]=U[f^*]$ and $U[g]=U[g^*]$). (2) is equivalent with

(2') $U[af+g]=S(U[f],U[g];a)$

[$S(x,y;a)$ is an *arbitrary* function of three variables].

This contains the case of linear operations, because then $S(x,y;a)= =ax+y$.

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1) F. RIESZ: Sur les opérations fonction lles linéaires, *Comptes Rendus* de l'Académie des Sciences, Paris **149** (1909), 974–977. — F. RIESZ: Démonstration nouvelle d'un théorème concernant les opérations fonctionelles lenéaires, *Annales de l'École Normale Supérieure* (3) **31** (1914), 9–14. — F. RIESZ: Untersuchungen über Systeme integrierbarer Funktionen, *Math. Annalen* **69** (1910), 440–497, see p. 475. — M. FRÉCHET Sur les ensembles de fonctions et les opérations linéaires, *Comptes Rendus de l'Académie des Sciences Paris* **144** (1907), 1414–1416. — M. H. STEINHAUS: Additive und s'e ige Funktionaloperationen, *Math. Zeitschrift* **5** (1918), 186–221.

2) G. H. HARDY–J. E. LITTLEWOOD–G. PÓLYA: Inequalities, Cambridge (1934) pp. 160–161. — There is a theorem of B. DE FINETTI (G. H. HARDY–J. E. LITTLEWOOD–G. PÓLYA, loc. cit. pp. 157–163) which can be transformed in a special case of our theorem for increasing functions with the range (0,1). (He supposes instead of the continuity the strict monotony of the operation and supposes (2) in a slightly altered form.). Our theorem — if we suppose also the monotony of U— might be reduced to that of DE FINETTI.

Theorem I.:[3]) *If and only if $U[f]$ is quasilinear, then there exists a continuous, strictly monotonic function $t(u)$ and a linear functional operation $L[f]$ on F such that*

(3) $$U[f]=t(L[f])$$

i. e. $U[f]$ can be transformed in a linear operation.

We can see from (3) that a solution of (2') exists only if the „arbitrary" functions S has the form

$$S(x,y;a)=t\{a.t^{-1}(x)+t^{-1}(y)\}$$

Proof: We can suppose that U is not identically constant; let $f_0 \in F$ denote a function for which $U[f_0] \neq U[0]$ We define

(4) $$t(u)=U[u.f_0].$$

It is evident, that $t(u)$ is continuous; but it is strictly monotonic too, because else there were $A \neq B$ such that $t(A)=t(B)$ i. e.

$$U[f_0]=U\left[\frac{1}{B-A}\cdot Bf_0+\frac{1}{A-B}\cdot Af_0\right]=U\left[\frac{1}{B-A}\cdot Af_0+\frac{1}{A-B}\cdot Af_0\right]=U[0]$$

in contradiction with the definition of f_0.

Let the range of $t(u)$ be (C,D): $C<t(u)<D$, then $C<t(0)=U[0]<D$. For an arbitrary $f \in F$ $\lim\limits_{v\to 0} U[v\cdot f]=U[0]$ by the continuity of U so that also for sufficiently small $u_1 \neq 0$ $C<U[u_1\cdot f]<D$ holds and thus there exists an u_0 such that $U[u_1\cdot f]=U[u_0\cdot f_0]$. Then

$$U[f]=U\left[\frac{1}{u_1}\cdot u_1 f\right]=U\left[\frac{1}{u_1}\cdot u_0 f_0\right]=t\left(\frac{u_0}{u_1}\right)$$

and so $C<U[f]<D$. Thus there exists a function t^{-1}, the inverse function of $t(u)$ which is, of course, also continuous and strictly monotonic and which has a sense for all $C<t<D$.

Now we can define $L[f]=t^{-1}(U[f])$. $L[f]$ is continuous and as $U[f]=t\,t^{-1}(U[f])=U[t^{-1}(U[f])\cdot f_0]$ we have

$$L[af+f_1]=t^{-1}(U[af+f_1])=t^{-1}(U[\{at^{-1}(U[f])+t^{-1}(U[f_1])\}\cdot f_0])=$$
$$=t^{-1}t(aL[f]+L[f_1])=aL[f]+L[f_1].$$

Thus $L[f]$ is linear and this completes our proof.

Our theorem remains true if we suppose besides the continuity of $U(f)$, instead of (2) only its „quasi-distributivity" that is continuity and

(5) $$U[f+g]=U[f^*+g] \text{ whenever } U[f]=U[f^*]$$

or what is the same:

(5') $$U[f+g]=S(U[f]\ U[g]).$$

[3]) The author of this paper proved this theorem originally only for strictly monotonic functionals. It was P. C. Rosenbloom who kindly called his attention to the fact, th t his methods of proof are sufficient to prove the theorem in its present, more general form.

Theorem II.: *If and only if $U[f]$ is quasidistributive, then there exists a continuous, strictly monotonic function $t(u)$ and a linear functional operation $L[f]$ on F such, that*

$$U[f]=t(L[f]).$$

Proof: $S(x\ y)$ is associative: $S\{S(x,y),z\}=S\{x,S(y,z)\}$ and if we write $x=U[f], e=U[0], x^{-1}=U[-f]$ then $S(e,x)=x$, $S(x^{-1},x)=e$. Thus $x=U(f)$ forms (with the „product" S) a one dimensional topological (continuous) group. It is well-known (see e. g. B. L. VAN DER WAERDEN: Über die Theorie der kontinuierlichen Gruppen, Göttingen, 1924, pp. 90–96) that such groups are isomorphic with the translation-group i. e. there exists a continuous, strictly increasing function $s(t)$, such that $S(x,y)=$ $=s^{-1}\{s(x)+s(y)\}$, [$s^{-1}(u)$ is the inverse function of $s(t)$], for all x,yu. Thus $sU(f+g)=sU(f)+sU(g)$, or with $t(u)=s^{-1}(u)$: $t[f]=t^{-1}(U[f])$ is linear and continuous; q. e. d. [Here (5') can be solved only if $S(x\ y)=$ $=t\{t^{-1}(x)+t^{-1}(y)\}$.]

We remark that in the proof of Theorem I we could give the *explicit form* of $t(u)$, while here we proved only its *existence* [though the proof of the theorem quoted above gives a *construction* for $t(u)$].

(P. C. ROSENBLOOM has obtained similar results supposing instead of (2) the weaker condition

$$U\left[\frac{f+g}{2}\right]=U\left[\frac{f^*+g}{2}\right] \text{ whenever } U[f]=U[f^*]$$

[similar to (5)] and working in more general functional spaces).

Using the theorems of F. RIESZ, M. FRÉCHET and M. H. STEINHAUS mentioned above we have the quasilinear (or quasiditributive) operations in the space of continuous functions, resp. in the L^p space as functions of Stieltjes integrals, resp. of Lebesque integrals with weight-functions.

(Received Octobre 9, 1950.)