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Two versions of graded rings

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Abstract. Let R be a ring which is a sum of its additive subgroups R_s, $s \in S$. Suppose that all rings among the R_s are nilpotent. We give new conditions on the interaction of the R_s sufficient for R to be nilpotent. Analogous results are obtained for locally nilpotent and quasi-regular rings.

1. Introduction

The motivation for this paper stems from the following general question. Let $\mathcal{K}$ be a class of rings, S a set, R an associative ring, and let $R = \sum_{s\in S} R_s$ be a sum of additive subgroups R_s of R. Suppose that all rings among the R_s belong to $\mathcal{K}$. It is only natural to ask whether it follows that R belongs to $\mathcal{K}$, too. The first result in this area is due to Kegel [8], who proved that R is nilpotent provided that $|S| = 2$ and both the R_s are nilpotent rings. Denote by $\mathcal{N}$, $\mathcal{L}$ and $\mathcal{J}$ the classes of all nilpotent, locally nilpotent, and quasi-regular rings, respectively. In general, the results of [8], [2], [6] and [12] show that for classes $\mathcal{N}$ (if $|S| > 2$) and $\mathcal{L}$, $\mathcal{J}$ (if $|S| > 1$) the answer to the question is negative even if one demands that all the R_s are subrings and so lie in the class considered.

This raises the interesting problem of what additional assumptions are needed to imply that $R = \sum_{s\in S} R_s$ belongs to $\mathcal{K}$, as soon as all rings among the R_s belong to $\mathcal{K}$.

The assumptions of this sort known in the literature (see, for example, [9], [1], [4]) impose a restriction on the interaction of the components R_s by requiring that S be a semigroup and R be an S-*graded* ring. (This means that R is a direct sum of the R_s, and $R_sR_t \subseteq R_{st}$ for any $s, t \in S$). Our present goal is to obtain new conditions sufficient for the positive answer to the general question above.

2. Main results

Let S be a semigroup, $R = \sum_{s\in S} R_s$ a sum of additive subgroups R_s of R. We say that R is a *structural S-sum* if and only if, for each subsemigroup (left ideal, right ideal) T of S, the sum $R_T = \sum_{t\in T} R_t$ is a subring (respectively, left ideal, right ideal) of R.

Evidently, every S-graded ring is a structural S-sum. Examples of structural S-sums which are not S-graded can be easily given with the use of AS-rings (cf. [13]). An important difference between S-graded rings and structural S-sums is the following. Suppose that T is a subsemigroup of S. If R is an S-graded ring, then obviously R_T is T-graded. However, if R is a structural S-sum, then R_T may be not a structural T-sum, because T may have new ideals which do not come from S.

The case where $|S| = 1$ is trivial, and so throughout we assume that S is not a singleton. A semigroup entirely consisting of idempotents is called a *band*.

Theorem 2.1. *The following are equivalent:*

(i) *for every structural S-sum $R = \sum_{s\in S} R_s$, if all rings among the R_s are nilpotent, then R is nilpotent;*

(ii) *S is a finite band.*

Theorem 2.2. *The following are equivalent:*

(i) *for every structural S-sum $R = \sum_{s\in S} R_s$, if all rings among the R_s are locally nilpotent, then R is locally nilpotent;*

(ii) *for every structural S-sum $R = \sum_{s\in S} R_s$, if all rings among the R_s are quasi-regular, then R is quasi-regular;*

(iii) *S is a band.*

Let S be a semigroup, $R = \sum_{s\in S} R_s$. If $X \subseteq S$, then we put $R_X = \sum_{s\in X} R_s$. For any $s \in S$ denote by $\langle s\rangle$ the subsemigroup generated in S by s, and put $R^s = R_{\langle s\rangle}$. We say that R is an *S-sum* if $R_sR_t \subseteq R^{st}$ for all $s, t \in S$.

Clearly, every S-graded ring is an S-sum and every S-sum is a structural S-sum. In the special case where S is a *semilattice*, i.e. a commutative band, the concept of an S-sum coincides with that of a semilattice sum (cf. [17]). Note that the preservation of various ring properties by semilattice sums was investigated in [17], [16], [3], and other papers.

Theorem 2.3. *The following are equivalent:*

(i) *for every S-sum $R = \sum_{s\in S} R_s$, if all rings among the R_s are nilpotent, then R is nilpotent;*

(ii) *S^0 has a finite ideal chain $0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S^0$ such that, for $i = 1, \dots, n$, factor S_i/S_{i-1} is finite or nilpotent, and all subgroups of S are 2-groups.*

Theorem 2.4. *The following are equivalent:*

(i) *for every S-sum $R = \sum_{s\in S} R_s$, if all rings among the R_s are locally nilpotent, then R is locally nilpotent;*

(ii) *S is locally finite and all subgroups of S are 2-groups.*

As to the analogous question on quasi-regular rings, we can give an answer in the special case of PI-rings.

Theorem 2.5. *Suppose that S is a locally finite semigroup such that all subgroups of S are 2-groups. Let $R = \sum_{s\in S} R_s$ be an S-sum with all subrings among the R_s being quasi-regular, and let R be a PI-ring. Then R is quasi-regular.*

We can show that the restrictions on S are necessary, and so can not be removed from Theorem 2.5. However, it is an open question whether it is possible to drop the condition that R be a PI-ring. This is connected to the following

Problem 2.1. *Does there exists a ring R which is not quasi-regular but is a sum of a quasi-regular subring E and an additive subgroup F such that $F^2 \subseteq E$?*

3. Proofs

For the standard concepts concerning semigroups and S-graded rings we refer to [5] and [14].

A band H is said to be *rectangular* if it satisfies the identity $xyx{=}x$. Every band B is a *semilattice Y of rectangular bands H_y*, $y \in Y$ (cf. [5], Exercise 1 in §4.2). This means that there exists a semilattice Y and paiwise disjoint rectangular bands H_y such that $B = \bigcup_{y\in Y} H_y$ and $H_y H_z \subseteq H_{yz}$ for all $y, z \in Y$. We shall use the natural partial order $\leq$ defined on Y by the rule $y \leq z \Leftrightarrow yz = y$.

Let $R = \sum_{s\in S} R_s$ be a structural S-sum. For any $r \in R$ we fix some expression of r in the form $r = \sum_{s\in S} r_s$ where $r_s \in R_s$ and there is only a finite number of $r_s \neq 0$. The set $\operatorname{supp}(r) = \{s \in S \mid r_s \neq 0\}$ will be called the *support* of r.

The following lemma was included in Shevrin's lectures on periodic semigroups at the Ural State University.

Lemma 3.1 (L. N. Shevrin). *Suppose that a periodic semigroup S contains only a finite number of idempotents, each nil factor of S is nilpotent, and every subgroup of S is finite. Then S^0 has a finite ideal chain $0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S^0$ such that, for $i = 1, \dots, n$, factor S_i/S_{i-1} is finite or nilpotent.*

Proof is contained in [11], Lemma 11.

Lemma 3.2 ([9]). *If S is a semigroup which is not locally finite, then there exists an S-graded ring $R = \bigoplus_{s\in S} R_s$ such that all subrings among the R_s have zero multiplication, but R is not quasi-regular.*

Lemma 3.3 ([9]). *If G is an infinite group, then there exists an S-graded ring $R = \bigoplus_{s\in S} R_s$ such that all R_s are rings with zero multiplication, but R is not nilpotent.*

Lemma 3.4. *If S contains a non-idempotent element, then there exists a structural S-sum $R = \sum_{s\in S} R_s$ such that all subrings among the R_s have zero multiplication, but R is not quasi-regular.*

Proof. Let $t \in S$, $t \neq t^2$. Take any field F and put $R = F[x]/(x^3)$, $R_t = F + Fx$, $R_{t^2} = Fx^2$. For any $s \in S \backslash \{t, t^2\}$, let $R_s = 0$. Then R has all the required properties.

Lemma 3.5. *If S has an infinite number of idempotents, then there exists an S-sum $R = \sum_{s\in S} R_s$ such that all the R_s are nilpotent rings, but R is not nilpotent.*

Proof. If S contains infinitely many idempotents $e_1, e_2, \dots$, then we can take any field F and put $R_{e_i} = xF[x]/(x^i)$, $R = \prod_{i=1}^{\infty} R_{e_i}$, where $R_s = 0$ for $s \notin \{e_1, e_2, \dots\}$, and get a contradiction, because R is not nilpotent but all R_s are nilpotent.

Lemma 3.6. *Let S be a finite semigroup, $R = \sum_{s\in S} R_s$ an S-sum, $\mathcal{K}$ one of the classes $\mathcal{N}$, $\mathcal{L}$, $\mathcal{J}$, and let $R_G \in \mathcal{K}$ for each subgroup G of S. Then $R \in \mathcal{K}$, too.*

Proof follows from [15], the proof of Lemma 4.1 (cf. [11], Lemma 5).

Lemma 3.7. *Let G be a group with an element g of odd order. Then there exists a G-sum $R = \sum_{s\in G} R_s$ such that all subrings among the R_s have zero multiplication, but R is not quasi-regular.*

Proof. Put $f = g^2$. Let $R[x, y]$ be the ring of polynomials over a field F in commuting variables x, y without free terms. Consider the ring $R = F[x, y]/I$, where I is the ideal generated by x^3, y^3, and xy. Put $R_f = Fx + Fy^2$, $R_g = Fy + Fx^2$, and $R_t = 0$ for $t \in G \backslash \{f, g\}$. Evidently, all subrings among the R_s are equal to zero. Besides, $R_f R_g = R_g R_f = 0$

and so these products do not contradict the definition of an S-sum. Since g^2 and f^2 generate the same subgroup in G as g, clearly R_g^2 and R_f^2 can be defined arbitrarily. Therefore R is a required example.

If G is a group with identity e, and $R = \sum_{g\in G} R_g$ is a G-sum, then R_e is called the *initial component.* A routine verification gives us

Lemma 3.8. *Let G be a group with a normal subgroup N, and let $R = \sum_{g\in G} R_g$ be a G-sum. Then $R = \sum_{gN\in G/N} R_{gN}$ is a G/N-sum with initial component R_N.*

Lemma 3.9. *Let $G = \{e, g\}$ be a group, and let $R = \sum_{g\in G} R_g$ be a G-sum. If R_e is nilpotent (locally nilpotent), then R is nilpotent (locally nilpotent), too.*

Proof. We shall record the proof only for local nilpotency. (The proof for nilpotency is analogous with a few simplifications). Take any finite set M of elements of R. We must prove that $M^d = 0$ for some d.

Denote by $H(R)$ the set of *homogeneous* elements of R, i.e. $H(R) = \bigcup_{g\in G} R_g$. Note that $H(R)$ may be not closed under multiplication. If $r \in H(R)$, then we fix an element $h(r)$ of G such that $r \in R_{h(r)}$. For each $x \in R$ we fix one expression of x as a sum of homogeneous elements. We can replace each $x \in M$ by all homogeneous summands in this expression, and assume that M is a finite set of homogeneous elements. Further, put $M_k = \bigcup_{i=1}^k M^i$. Put $P_1 = M$, and if $i > 1$ then denote by P_i the union of P_{i-1} and the set of the homogeneous summands of elements of $MP_{i-1} \cup P_{i-1}M$. Clearly, all P_i are finite.

Denote by m the nilpotency index of the subring generated in R_e by $R_e \cap M$. Let n be the nilpotency index of the subring generated in R_e by $R_e \cap P_{m+1}$. Put $d = n(m+1)$. We claim that $M^d = 0$.

In order to prove this, we introduce auxilliary sets I_i^j where $i, j \geq 0$, and will show that

$$M^d \subseteq I_0^d R^1 \subseteq I_1^{d-m-1} R^1 \subseteq I_2^{d-2m-2} R^1 \subseteq I_n^0 R^1 = 0.$$

For $i, j \geq 0$, denote by I_i^j the set of all sums of products $x_1 \dots x_i y_1 \dots y_j$ such that $x_1, \dots, x_i \in R_e \cap P_{(m+1)}$, and $y_1, \dots, y_j \in M$.

First we check that $I_i^{j(m+1)} \subseteq I_{i+1}^{(j-1)(m+1)} R^1$ for all $i \geq 0, j \geq 1$. Pick any product

$$p = x_1 \dots x_i y_1 \dots y_{j(m+1)} \in I_i^{j(m+1)}.$$

and consider several cases.

Case 1. $h(y_1) = e$. Then the claim is obvious, because $y_1 \in R_e \cap M$ implies $p \in I_{i+1}^{(j-1)(m+1)} R^1$ by the definition of $I_{i+1}^{(j-1)(m+1)}$.

Case 2. $h(y_1) = g$ and $h(y_1)h(y_2) = e$. Then $y_1y_2 \in R_e \cap P_2 \subseteq R_e \cap P_{m+1}$, and the claim is clear as in Case 1.

Case 3. $h(y_1) = g$, $h(y_1)h(y_2) = g$, and $h(y_1)h(y_2)h(y_3) = e$. Then $h(y_2) = e$ and $h(y_3) = g$. Clearly, $y_1y_2 = x + z$, where $x \in R_e$, and $z \in R_g$. Therefore $p = u + w$, where $u = x_0 \dots x_i x y_3 \dots y_j$ and $w = x_0 \dots x_i z y_3 \dots y_j$. Since $u \in I_{i+1}^{(j-1)(m+1)} R^1$, it remains to show that $w \in I_{i+1}^{(j-1)(m+1)} R^1$. However, $h(z)h(y_3) = gg = e$, and so the claim follows as in Case 2.

Case 4. $h(y_1) = \dots = h(y_{i-1}) = g$ and $h(y_1)\dots h(y_i) = e$, for some $i \in \{1, \dots, m+1\}$. This case is similar to Case 3, only we use $h(y_1)\dots h(y_{i-1}) = x + z$, where $x \in R_e$, and $z \in R_g$.

Given that $G = \{e, g\}$, it remains to consider the following

Case 5. $h(y_1) = h(y_1)h(y_2) = \dots = h(y_1)\dots h(y_{m+1}) = g$. Then $h(y_2) = \dots = h(y_{m+1}) = e$, and by the choice of m we get $y_2 \dots y_{m+1} = 0$. So $p = 0 \in I_{i+1}^{(j-1)(m+1)} R^1$, as claimed.

Lemma 3.10. *Let G be a finite 2-group, and let $R = \sum_{g\in G} R_g$ be a G-sum. If R_e is nilpotent, then R is nilpotent, too.*

Proof. Every finite 2-group is nilpotent, and so has a central series with factors of order 2. Therefore the proof follows from Lemmas 3.8 and 3.9.

Lemma 3.11. *Let G be a locally finite 2-group, $R = \sum_{g\in G} R_g$ a G-sum. If R_e is locally nilpotent, then R is locally nilpotent, too.*

Proof. Take any finite set M of elements of R. We must prove that $M^d = 0$ for some d. Without loss of generality we may assume that G is the subgroup generated by the supports of all elements in M. Thus we assume that G is finite. Lemma 3.10 completes the proof.

The following lemma is due to Grzeszczuck [7], the proof of Theorem 1.

Lemma 3.12. *Let G be a finite group satisfying the identity $x^2 = e$, and let $R = \sum_{g\in G} R_g$ be a G-sum with unity 1. Then $1 \in R_e$.*

Proof. For any $s, t \in G$, it follows that $\{st\} \cup \{e\}$ is a subgroup, because G satisfies $x^2 = e$. The definition of a G-sum yields $R_sR_t \subseteq R_{st} + R_e$. Therefore, for every two non-empty *subsets* $S, T \subseteq S$, we get $R_SR_T \subseteq R_{ST} + R_e$. Using this we can repeat the proof of Theorem 1 in [7], and show by induction on $|G\backslash S|$ that if $e \in S$ then $1 \in R_S$. If we take the set $\{e\}$ as S, then we get $1 \in R_e$, as required.

Lemma 3.13. *Let G be a finite 2-group, and let $R = \sum_{g\in G} R_g$ be a G-sum with unity 1. Then $1 \in R_e$.*

Proof follows from Lemmas 3.12 and 3.8, because every finite 2-group has a central series with factors of order 2.

Proof of Theorem 2.1. Suppose that S satisfies (i). Then Lemma 3.4 implies that S is a band. By Lemma 3.5 S is finite. Thus (i) implies (ii).

Suppose that S is a finite band. Take any structural S-sum $R = \sum_{s\in S} R_s$ such that every subring among the R_s is nilpotent. Since S is a band, every element s of S forms a one-element subsemigroup in S, and hence R_s is a subring. Therefore all R_s are nilpotent. Let S be a semilattice Y of rectangular bands H_y. For $y \in Y$ put $Q_y = \sum_{s\in H_y} R_s$. Let $\overline{Y} = \{y \in Y \mid Q_y \neq 0\}$. We shall show by induction on $|\overline{Y}|$ that R is nilpotent.

The case where $|\overline{Y}| = 0$ is trivial. Now suppose that $|\overline{Y}| \geq 1$, pick a minimal element z in $\overline{Y}$, and put $H = H_z$. By the choice of z we get $R_H = R_{SHS}$. Hence R_H is an ideal of R, because SHS is an ideal of S.

Take any $h, g \in H$. It is routine to verify that $H \cap hS = hH$ and $H \cap Sg = Hg$, because H is one of the rectagular components of S. Since hS and Sg are right and left ideals of S, respectively, the definition of a structural S-sum yields $R_hR_g \subseteq R_{hS} \cap R_{Sg} \cap R_H = R_{hH} \cap R_{Hg} = R_{hg}$, because H is a rectangular band. It follows that R_{hH} is a sum of its nilpotent left ideals $R_{hg}, g \in H$. Hence R_{hH} is nilpotent. Further, R_H is also nilpotent, because it is a sum of its nilpotent right ideals R_{hH}, $h \in H$. Now we can factor out ideal R_H, and pass to the ring $R/R_H = \sum_{s\in S}(R_s + R_H)$ which is nilpotent by the induction assumption. This completes the proof.

Proof of Theorem 2.2. Implications (i) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iii) follow from Lemma 3.4.

Suppose that S is a band. Denote by $\mathcal{F}$ the set of all finite subbands of S. Take any structural S-sum $R = \sum_{s\in S} R_s$ such that all the R_s are locally nilpotent. If $B \in \mathcal{F}$, then R_B is a subring of R. Although R_B may be not a structural B-sum, the same (induction on $|\overline{Y}|$) argument as in the proof of Theorem 2.1, using the fact that R_B is a structural S-sum, shows that R_B is locally nilpotent. It is known that every band is locally finite. Hence each finite subset of R is contained in R_B for appropriate finite B. Therefore R is locally nilpotent. Thus (iii) implies (i). Implication (iii) $\Longrightarrow$ (ii) is similar.

Proof of Theorem 2.3. Implication (ii) $\Longrightarrow$ (i) follows from Lemma 3.1 by induction on the length n of the ideal chain of S with the use of Lemma 3.10.

(i) $\Longrightarrow$ (ii). It follows from Lemma 3.7 that every subgroup of S is a 2-group. By Lemma 3.3 every subgroup of S is finite. Lemma 3.5 shows that S contains a finite number of idempotents.

If S has ideals $I \subseteq J \subseteq S$ such that J/I is nil but not nilpotent, then we can consider a contracted semigroup ring $R = F_0[J/I]$ over a field F. Clearly R is not nilpotent. However, if we put $R_j = F_j$ for $j \in J \backslash I$, and $R_s = 0$ for $s \notin J \backslash I$, then R becomes an S-sum with all subrings among the R_s equal to zero. This contradiction and Lemma 3.1 show that (ii) holds, completing the proof.

Proof of Theorem 2.4. Implication (i) $\Longrightarrow$ (ii) follows from Lemmas 3.2 and 3.7.

Lemmas 3.6 and 3.11 prove implication (ii) $\Longrightarrow$ (i) for a finite S. The case of a locally finite S follows immediately, as in the proof of Theorem 2.2 or Lemma 3.11.

Proof of Theorem 2.5. As in the proof of Theorem 2.4, Lemma 3.6 shows that it suffices to consider the case of a locally finite 2-group. As in the proof of Lemma 3.11, it remains to deal with the case where S is a finite 2-group. Denote the identity of the group S by e.

Suppose to the contrary that there exists an S-sum $R = \sum_{s \in S} R_s$ with quasi-regular component R_e, such that R satisfies a polynomial identity but is not quasi-regular. Let R/I be a primitive homomorphic image of R. Then it is readily verified that $R/I = \sum_{s \in S}(R_s + I)$ is an S-sum. Kaplansky's theorem implies that R/I has a unity 1. By Lemma 3.13, $1 \in R_e$. However, R_e is quasi-regular. This contradiction completes the proof.

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