On Zorn's lemma.

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In a lecture on infinite abelian groups I deduced ZORN's lemma[1]) in a simple way directly from the axiom of choice, without first proving ZERMELO's theorem on well-ordering. My proof is based on the fundamental idea of Zermelo's first proof[2]) of his theorem. H. KNESER has also given in a recent paper a very elegant proof of Zorn's lemma directly from the axiom of choice.[3]) My proof, however, seems to be somewhat shorter and simpler.

In his "Lattice Theory" G. BIRKHOFF[4]) shows by masterly arranged proofs not only Zermelo's theorem and Zorn's lemma but also two other important statements (see later *c*) and *d*)) to be equivalent to the axiom of choice. In Birkhoff's treatment, however, Zorn's lemma follows from the axiom of choice only with the intermediation of Zermelo's theorem, for which Zermelo's first proof is reproduced. It seems to me therefore worth the trouble to complete my proof and give by this way an even shorter proof for the equivalence of the five statements figuring at Birkhoff.

A simply ordered subset C of a partly ordered set P is said to be a *chain*. An element $u \in P$ is an *upper bound* of a subset T of P if $t \leqq u$ for every $t \in T$. An element m of P is *maximal* in P if in P there exists no element $p > m$. A subset M of a set S having a certain property is *maximal* in S if there exists no subset of S of this property which contains M as a proper subset. A property Φ defined for some subsets of a set S is called *a property of finite character* if the fact that a subset T of S has the property Φ (denoted: $T \in \Phi$) is equivalent to $F \in \Phi$ for every finite subset $F \subseteq T$.

The five equivalent statements are:

a) *Axiom of choice.* For an arbitrary set S there exists a single-valued function ε which selects from each non-void subset T of S a well-defined element $\varepsilon(T) \in T$.

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1) M. ZORN, A remark on method in transfinite algebra. *Bull. Amer. Math. Soc.* **41** (1935), 667–670.

2) E. ZERMELO, Beweis, daß jede Menge wohlgeordnet werden kann. *Math. Annalen* **59** (1904), 514–516.

3) H. KNESER, Eine direkte Ableitung des Zornschen Lemmas aus dem Auswahlaxiom. *Math. Zeitschrift* **53** (1950), 110–113.

4) G. BIRKHOFF, Lattice Theory. *Amer. Math. Soc. Coll. Publ.* **25.** Rev. Ed., 1948.

b) Zorn's lemma. If every chain of a partly ordered set P has an upper bound in P, then P contains a maximal element.

c) Tukey's lemma.[5]) If Φ is a property of finite character defined for some subsets of a set S, then there exists a maximal subset of S having the property Φ.

d) Birkhoff's theorem[5]) Every partly ordered set has a maximal chain.

e) Zermelo's theorem. Every set can be well-ordered.

Proof of the equivalence of the statements a)—e):

1. *a) implies b).* Let P be a partly ordered set satisfying the condition in Zorn's lemma, and ε a selecting function in the sense of *a)* defined for every subset of P. A chain H of P will be called *a principal chain* if H is well-ordered (by the ordering in P) and, for every $h \in H$, h is the element selected by ε from the set of all the elements of P which are greater than every element of H preceding h. With signs:

$$(*) \qquad h = \varepsilon(\underset{x}{\mathbf{E}}[x \in P; x > y \text{ for all } y \in H \text{ and } y < h]).[6])$$

We are now going to show that the union H^* of all principal chains of P is itself a principal chain.

First we prove that H^* is a chain. For let h_1, h_2 be two arbitrary elements of H^* and $h_1 \in H_1, h_2 \in H_2$, H_1, H_2 being principal chains. It is sufficient to show that one of H_1 and H_2 is a subset of the other. In the contrary case both of H_1 and H_2 would have a first element which is not contained in the other. But this is impossible, since every element of a principal chain H is uniquely determined according to (*) by the set of the preceding elements of H.

The chain H^*, moreover, is well-ordered. For let us consider a non-void subset K of H^* and an element k of K; the latter belongs to a principal chain H. Then k and the elements of K preceding k form a non-void subset of H which contains a first element. This is obviously the first element in K too.

Finally H^* is a principal chain, for every element of H^* belongs to a principal chain and so satisfies the condition (*).

Thus H^* is a principal chain in P which contains every principal chain of P.[7]) An upper bound of the chain H^* is obviously *a maximal element in P*, for in the contrary case P would contain at least one element greater than each element of H^* and so according to (*) a principal chain could be obtained which contains H^* as a proper subset.

[5]) Cf. G. Birkhoff, op. cit. p. 42.

[6]) (*) implies that $\varepsilon(P)$ is the first element of each principal chain.

[7]) The proof of this fact is simpler than that of the corresponding fact (concerning Γ-sequences) in Zermelo's first proof. The reason for that is that we are dealing with an *a priori* ordered set, while at Zermelo the ordering of the set is induced only during the proof.

2. *b) implies c).*[8]) Consider an arbitrary set S, for some subsets of which a property Φ of finite character is defined. The subsets of S with the property Φ form a set P partly ordered by the inclusion relation. By Zorn's lemma we have only to show that every chain of P has an upper bound in P, i. e. the union U of a chain of subsets $X_\nu \in \Phi$ of S is itself a set having the property Φ. This is true, for if $F = \{f_1, \ldots, f_n\}$ is any finite subset of U, then, each f_i belonging to some X_i, the largest of these X_i contains F. Hence $F \in \Phi$ and so $U \in \Phi$.

3. *c) implies d).*[9]) This is clear since the property of "being a chain" is of finite character.

4. *d) implies e).* Let S be an arbitrary set. Consider the set W of all the well-ordered subsets of S. (We emphasize that each subset of S, which can be well-ordered, is to be considered in each possible well-ordering as a different element of W.) We define $A < B$ for $A, B \in W$, if and only if the well-ordered subset A of S is an initial interval of B. Thus W becomes a partly ordered set which, by *d*), contains a maximal chain M consisting of the elements W_ν of W. On the union U of the subsets W_ν of S, the well-ordering of the W_ν-s induce a uniquely defined well ordering. Since the chain M is a maximal one, U cannot be a proper subset of S, i. e. $U = S$.

5. *e) implies* obviously *a*).

If only the equivalence of *a*), *b*) and *e*) is to be proved, **2** and **3** may be omitted.

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[8]) Cf. G. Birkhoff, op. cit. pp. 42—43.

[9]) Cf. G. Birkhoff, op. cit. p. 43.