Geometry in abstract distance spaces.*)

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Dedicated to my parents, Roy and Frances Ellis.

1. The literature on geometry in abstract distance spaces (sets bearing distances which are not, in general, real or complex numbers) is quite scattered and some of it is not readily available. An example of the latter is the work appearing by MENGER and TAUSSKY in *Ergebnisse eines Mathematischen Kolloquiums, Wien.* Even when one bears in mind the fact that this field is relatively undeveloped, a surprising number of mathematicians are unfamiliar with it or even unaware of its existence. This is certainly in contrast to the popularity of geometry in semimetric and metric spaces (this popularity will undoubtedly be increased by the forthcoming textbook on the subject by Professor L. M. BLUMENTHAL) which naturally leads to more abstract distance geometry. It is an unfortunate state of affairs since there appear to be many intimate links between abstract distance spaces and the topics of modern algebra and topology.

It is hoped that the discussion given in this paper will, to some degree, remedy these misfortunes. First we shall give a foreword on algebra and topology and an introduction showing how the most fundamental notions of geometry in abstract distance spaces are immediately obtainable as generalizations of the corresponding notions in the geometry of semimetric and metric spaces. Later we shall examine other geometric notions applicable to abstract distance spaces some of which seem of even more interest when presented in the abstract fashion than in the classical case. Finally, we shall give a survey of the distance-theoretic results obtained up to the present for the special types of abstract distance spaces which have been studied. In this part we have tried to mention the persons responsible for results of importance.

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The terminology connected with semimetric and metric spaces begins, of course, with FRÉCHET, HAUSDORFF and MENGER. Much of the terminology of the present author derives from that employed by L. M. BLUMENTHAL.

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2. List of special symbols. In this paper we employ the following logical symbols most of which are due to E. H. MOORE:

$\exists$ is read "there exist(s)"
$\exists|$ is read "there exist(s) uniquely"
$\cdot)\cdot$ or $:):$ is read "implies"
$\cdot\sim\cdot$ is read "if and only if"
$\wedge$ is read "and"
$\vee$ is read "or (conjunctive)"
$\dotplus$ is read "or (disjunctive)"
$\ni$ is read "such that"
$\forall$ is read "for all".

Set theoretic cross cut, union, and inclusion (in the wide sense) are denoted by $A \cap B$, $A \cup B$, and $A \subset B$, respectively. Class membership is denoted by $a \in A$ (read "a is an element of A"). The symbol $\{x|P\}$ denotes the set of x having property P.

Negation of a relation is denoted by a stroke as in $A \not\subset B$ (read "A is not part of B").

3. Foreword on algebra and topology. In this foreword we define briefly and in a unified fashion some of the standard terms from algebra and topology which appear in the following discussion.

A *composition set* C is a set admitting one or more (not necessarily a finite number) of *operations*, where by an operation we understand any function on C^k to C^j with possibly auxiliary variables from sets $A_1 A_2 \dots A_i$ involved (i, j, k any cardinals). This definition of composition set includes all of the familiar systems of modern algebra and topology (groupoids, vector spaces, topological spaces, lattices, etc.). An operation on C^k to C^j which does not involve auxiliary variables will be termed a k-ary, j-valued operation (when $k=2$, $j=1$, the operation is called a binary single-valued operation). If C admits only operations for which $j=1$ and k is finite, then C is called an *algebra*. (This is a very broad definition of algebra and should not be confused with the concept of algebra employed in structure theory as a vector space with ring multiplication over a division ring or field.)

Consider an algebra G admitting a binary single-valued operation $a*b$. Such an algebra is called a *groupoid*. A groupoid is *commutative, associative,* or *idempotent* according as $a*b=b*a$, $a*(b*c)=(a*b)*c$, or $a*a=a$

$(\forall a, b, c \in G)$, respectively. If $\exists e \in G \ni e*a = a*e = a$ $(\forall a \in G)$, then e is called a *unit element* for $a*b$. A unit element in a groupoid is unique when existent. An associative groupoid is usually referred to as a *semigroup*. If G has the property that when any two elements in $a*b=c$ are given the third is uniquely determined, then G is called a *quasigroup*. A quasigroup with a unit element is called a *loop*. A loop which is also a semigroup is called a *group*. One may show that a quasigroup which is also a semigroup is a *group*. We denote the operation in a group by $a+b$ and the unit element by 0. It is easily shown that if G is a group $a \in G \cdot) \cdot \exists | -a \in G$, $a+(-a) = = (-a)+a = 0$. One writes $a-b$ for $a+(-b)$. A group which is commutative is usually called an *abelian group*. If G is a group and H is a subset of G which is itself a group with respect to the operation of G, then H is called a *subgroup* of G. Let H be a subgroup of G and $a \in G$. The set $\{ax | x \in H\}$ is called a (left) *coset* of H. The *index* of H in G is the cardinal number of the class of mutually pairwise exclusive (left) cosets of H. The *order* or *period* of an element of a group G is the smallest positive integer n for which the "sum" of n of the given elements is 0; that is, $na = a+a+\ldots+a$ (n "summands") $= 0$. A mapping of the type $x' = x+a$ of a group onto itself is called a (right) *translation* by a. A mapping $x' = a-x$ is called a *reflection* in a. A group is called *cyclic* if it consists exclusively of the "multiples" (by positive integers) of a single element.

Let A be an algebra admitting two binary single-valued operations, $a+b$ and ab. One says that ab is *distributive* over $a+b$ if $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ $(\forall a, b, c \in A)$. If A forms an abelian group under $a+b$ and ab is distributive over $a+b$ and A is associative with respect to ab, then A is called a *ring*. A ring is *commutative* or is a *ring with unity* according as it is commutative with respect to ab or has a unit element 1 with respect to ab (of course, an arbitrary ring is not necessarily either commutative or a ring with unity). A ring with unity for which $ab = 0 \cdot) \cdot a = 0 \vee b = 0$ is called a *division ring*. A commutative division ring is called an *integral domain*. A commutative ring with unity whose elements other than 0 form a quasigroup under ab is called a *field*

Let L be an algebra admitting two binary single-valued operations, $a \vee b$ and $a \wedge b$. If L is associative, commutative, and idempotent with respect to both operations and if $a \vee b = a \cdot \sim \cdot a \wedge b = b$, then L is called a *lattice*. A *partial ordering* (binary, reflexive, antisymmetric, transitive relation) in which l. u. b. and g. l. b. exist for arbitrary finite subsets may be introduced into a lattice by defining $a \leq b \cdot \sim \cdot a \vee b = b$. If a lattice is such that one operation distributes over the other, it is easy to show that the second operation also distributes over the first. Let L be a lattice with unit elements 0 for $a \vee b$ and 1 for $a \wedge b$. If $x \in L \cdot) \cdot \exists x' \in L \ni x \wedge x' = 0 \wedge x \vee x' = 1$, then L is called *complemented*. A distributive, complemented lattice is called a *Boolean algebra* and it is seen that in a Boolean algebra, complements (of elements) are unique.

We next consider the introduction of a topology by methods closely resembling those of algebras. Let T be a set and $\mathfrak{n}$ be an element formally distinct from the elements of T. Suppose that $T^* = T \cup (\mathfrak{n})$ is a composition set admitting an $\aleph_0$-ary, single-valued operation, $\lim_{n \to \infty} a_n$. Then one says that a *topology* is defined in T by $\lim_{n \to \infty} a_n$. If a_n is a sequence of elements of T it is called *convergent* or *non convergent* according as $\lim_{n \to \infty} a_n \in T$ or $\lim_{n \to \infty} a_n = \mathfrak{n}$, respectively. T is called a *topological space*. If $A \subset T$ and $\exists\, a_n \in A\, (\forall n) \ni a_i \neq a_j\ (i \neq j) \wedge \lim_{n \to \infty} a_n = a \wedge a \in T$, then a is called an *accumulation element* of A. The set of all accumulation points (elements) of A is called the *derived set* of A and the set-theoretic sum of A and the derived set of A is called the (topological) *closure* of A. The closure of A is written $\bar{A}$. A is *closed* if $A = \bar{A}$. A is *open* if the set-theoretic complement of A in T is closed. A is *dense* in $B \subset T$, if $B \subset \bar{A}$. Various other topological notions (connectedness, local connectedness, etc.) may be defined in terms of open and closed sets.

The other usual methods (other than MOORE-SMITH convergence) of defining a topology may be accomplished by the preceding method provided the axiom of choice is allowed. Of course, strong restrictions must be placed on the operation $\lim_{n \to \infty} a_n$ in order to reach the more familiar types of topological spaces (FRÉCHET class L, HAUSDORFF spaces, KURATOWSKI spaces, metric spaces, etc.).

4. Introduction. Let S and G be abstract sets and suppose there is a single-valued function $d(x, y)$ on S^2 to G. S is then called a *distance space* over G and G is called the *ground set* of the space S. (We are guided in this terminology by the terminology associated with normed linear spaces.) The function $d(x, y)$ is called the *distance function* or, more simply, *distance* in S. A ground set G and the set of all[1]) distance spaces over G constitute the *distance space category* over G. We denote the category of all distance spaces over G (subject to the restriction noted in footnote [1])) by $\mathfrak{D}(G)$. An arbitrary ground set and its distance space category form a basis for distance geometry in its most general form. As one would expect, however, no results of interest may be obtained without further assumptions (that is, no results of interest for distance geometry). It is necessary, in general, to restrict both the ground set and the "allowable" type of distance function in order to

[1]) Of course, a restricted domain of discourse must be chosen whose subsets are the point sets of elements of $\mathfrak{D}(G)$ in order to avoid the familiar paradoxes of set theory. This is due to the obvious fact that any given set may be made into a distance space over any given non-null ground set.

obtain desirable results. If the ground set G is fixed, then restrictions on the "allowable" type of distance function obviously yield subcategories of $\mathfrak{D}(G)$. In certain of these subcategories over a properly chosen ground set, many of the familiar notions of the geometry of semimetric and metric spaces may be generalized in a straightforward fashion.

Many of the most interesting problems in distance geometry are concerned with those elements of $\mathfrak{D}(G)$ whose point sets are G itself. Such spaces we shall call *ground spaces*. Thus, most of the "characterization problems" are concerned with characterizing, by means of distance-theoretic notions, the ground set bearing a suitable distance function among other distance spaces over the ground set. Also, the ground set often bears an algebraic structure and the inter-relations between the algebraic and distance-theoretic properties are of interest.

Throughout the paper, $\mathfrak{R}$, $\mathfrak{C}$, and $\mathfrak{R}^*$, denote the real field, the complex field, and the set of non-negative real numbers, respectively.

The fundamental notions appearing in the geometry of semimetric and metric spaces may be found in the original papers of KARL MENGER [20] [2]) and in L. M. BLUMENTHAL's book [4].

5. Congruence and congruence classes. Two elements S and S^* of $\mathfrak{D}(G)$ are said to be *congruent*, written $S \approx S^*$, provided there is a biuniform mapping (biuniform correspondence), $f\colon S \to S^*$, of S *onto* S^* so that for $x, y \in S$ we have $d(x, y) = d(f(x), f(y))$ [3]). Congruence is an equivalence relation (that is, a reflexive, symmetric, transitive, binary relation) and divides the elements of any subcategory of $\mathfrak{D}(G)$ into *congruence classes*. If it is desired to specify the particular mapping f which establishes congruence between S and S^* (and which is called a congruence between S and S^*) we write $S \approx S^*(f)$ or $S^* \approx S(f^{-1})$. The *distance geometry* associated with any subcategory of $\mathfrak{D}(G)$ is the study of those properties common to all elements of a congruence class in the given subcategory; that is, the study of *congruence invariants* if we wish to emphasize the mapping aspect of congruence rather than the equivalence class aspect. Of course, this is a very strict definition of distance geometry in accordance with KLEIN's Erlanger program, and it is usual to include under the heading of distance geometry other properties of distance spaces which are associated with congruence invariants.

6. Distance sets, generalizations of congruence. If $S \in \mathfrak{D}(G)$, the *distance set*, $D(S)$, of S is the set $\{x \in G \mid \exists\, a, b \in S \ni d(a, b) = x\}$. This notion, of much intrinsic interest, is useful in formulating a certain generalization of the notion of congruence as is done in the next paragraph.

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[2]) The numbers in brackets refer to the bibliography at the end of this paper.

[3]) We use the same functional notation for the distance functions in S and S^* since these are distinguished by the context or by their arguments.

There have been two generalizations of congruence suggested, in addition to the possibility of similarity mappings between distance spaces over ground sets which are not necessarily identical, and we examine these now. L. M. BLUMENTHAL has suggested what we term *multi-valued congruence* as follows: Let $S, S^* \in \mathfrak{D}(G)$ and suppose there is an exactly 1-to-k mapping of S onto S^*, $f: S \overset{k}{\to} S^*$, with the property:[4]).

$$x, y \in S \cdot) \cdot \exists x' \in f(x), y' \in f(y) \ni d(x, y) = d(x', y').$$

An example of multi-valued congruence in the case $k = 2$ is given by the point to diametral point pair mapping of the $E_{2,r}$ (2-dimensional elliptic space of space constant r) onto the $S_{2,r}$ (convexly metrized two-sphere of radius r).

The writer has suggested the notion of *distanciallity* as follows: Two elements $S, S^* \in \mathfrak{D}(G)$, are said to be *distancial*, written $S \approx S^* \pmod D$, provided $D(S) = D(S^*)$. The notions of distanciallity and of multi-valued congruence compare as follows.

1. Multi-valued congruence replaces the single-valued mapping of congruence with a multiple-valued mapping while distanciallity is completely free from the notion of mapping and, hence, from any *a priori* restrictions on the cardinal numbers of spaces considered. (That is, spaces with different cardinal numbers may be distancial).

2. Distanciallity is again an equivalence relation while multi-valued congruence is non-symmetric, in general, for $k \neq 1$. Thus, one might say that, multi-valued congruence is a set-theoretic generalization and distanciallity is a primarily algebraic generalization of congruence since the subject of equivalence relations and equivalence classes is of much importance in modern algebra.

One may also define a relation of *similarity* between $S \in \mathfrak{D}(G)$ and $S^* \in \mathfrak{D}(G')$ as follows: Let $f: S \to S^*$ be a single-valued mapping of S onto S^* with the property.

If $a, b, c, d \in S$ then $a, b \approx c, d \cdot) \cdot f(a), f(b), \approx f(c), f(d)$. Then S and S^* are called *similar* distance spaces and the mapping f is called a *similitude*.

These three generalizations of congruence give rise to three geometries in abstract distance spaces. The first two, which we shall call 1-to-k geometry and distanciallity geometry, respectively, relate to a given distance space category, while the last, similarity geometry, applies to any class of distance spaces. None of these generalizations will be considered further in this paper.

7. Superposability properties and the group of motions. Let $S \in \mathfrak{D}(G)$. A *motion* of S is a congruence of S with itself. Two subsets S' and S^* of S are said to be *superposable*, written $S' \simeq S^*$, provided

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[4]) See the list of symbols in Section **2.**

there is a motion of S mapping S' onto S^*. Superposability is an equivalence relation in the Boolean algebra of subsets of S (the subsets of any set form a Boolean algebra under the set-theoretic operations). Clearly, $S' \simeq S^* .) \cdot S' \approx S^*$. However, the proposition $S' \approx S^* .) \cdot S' \simeq S^*$ is not, in general, valid. S is said to have the *property of k-point superposability* provided that if S' and S^* are subsets of S having k points each and $S \approx S^*$, then $S' \simeq S^*$. S is said to have the property of *free mobility* provided that for any two subsets, S' and S^*, of S, $S' \approx S^* (f)$ implies $S' \simeq S^* (g)$, where f and g agree on S'; that is, for any congruent mapping of one subset of S onto another there is a motion of S which induces the original congruence between these subsets.

The motions of S obviously form a subgroup of the group of point-permutations of S. It is referred to as the *group of motions* of S and written $\mathfrak{G}(S)$. $\mathfrak{G}(S)$ is said to be *simply transitive, k-tuple congruence transitive*, or *completely congruence transitive*, according as S has the property of 1-point superposability, k-point superposability, or free mobility, respectively.

8. Congruent imbedding. As remarked in **5,** the relation of congruence divides the elements of any subcategory of $\mathfrak{D}(G)$ into congruence classes. It is frequently of interest to determine conditions under which elements of a subcategory of $\mathfrak{D}(G)$ will lie in the congruence class of a given space or will lie in a congruence class with a subset of a given space. These two problems are described (after BLUMENTHAL) as the *space problem* and the *subset problem*, respectively.

Lat $S, S^* \in \mathfrak{D}(G)$. S is said to be *congruently contained* in S^* or merely (congruently) *imbeddable* in S^* provided S is congruent to a subset of S^*. I S is imbeddable in S^* we write $S \Subset S^*$.

Let $S \in \mathfrak{D}(G)$ and $\mathfrak{K}(G) \subset \mathfrak{D}(G)$. A solution of the space problem for S in $\mathfrak{K}(G)$ is given by conditions, expressed only in terms of distances and notions derivable from distances, which are necessary and sufficient in order that any given element of $\mathfrak{K}(G)$ which satisfies these conditions be congruent with S. A solution of the subset problem for S in $\mathfrak{K}(G)$ is given by conditions, expressed only in terms of distances and notions derivable from distances, which are necessary and sufficient in order that any given element of $\mathfrak{K}(G)$ which satisfies these conditions be congruently contained in S.

It is, of course, possible to solve the space problem for S by solving the subset problem for S in the given subcategory and then characterizing S (in terms of distances) among its subsets. This was, in fact, the approach to the space problem for E_n (n-dimensional Euclidean space) among semimetric spaces as originally solved by KARL MENGER [20].

Of aid in studying the subset problem are the notions of congruence indices and congruence order. Let $S \in \mathfrak{D}(G)$ and $\mathfrak{K}(G) \subset \mathfrak{D}(G)$. S is said to have *congruence indices* (n, k) in $\mathfrak{K}(G)$ provided that any element of $\mathfrak{K}(G)$

having more than $n+k$ pairwise distinct points is imbeddable in S whenever each n of its points are congruently contained in S. A space having congruence indices $(n, 0)$ $((n, 1))$ in $\mathfrak{K}(G)$ is said to have *congruence order n (quasi-congruence order n)* in $\mathfrak{K}(G)$. (In these definitions k and n are any cardinals). If S has congruence order n in $\mathfrak{K}(G)$ and does not have this property for any cardinal less than n then n is called the *best congruence order* of S in $\mathfrak{K}(G)$. The obvious value of having a congruence order of S in $\mathfrak{K}(G)$ is that the subset problem for S in $\mathfrak{K}(G)$ is reduced to finding conditions under which subsets of cardinal not exceeding the congruence order of elements of $\mathfrak{K}(G)$ are imbeddable in S. It is also convenient to make the special definition: S has *hyperfinite congruence order* in $\mathfrak{K}(G)$ provided S does not have any finite congruence order in $\mathfrak{K}(G)$ but any element of $\mathfrak{K}(G)$ is congruently contained in S whenever each of its finite subset are imbeddable in S.

The totality of space and subset problems for the elements of a distance space category is referred to as the class of characterization problems for the category. The class of characterization problems for the category of semimetric spaces has been the subject of a great deal of the literature on classical distance geometry [4].

9. Restrictions on "allowable" distance functions. We now obtain subcategories of $\mathfrak{D}(G)$ by considering those elements of $\mathfrak{D}(G)$ whose distance functions satisfy further restrictions. Of course, these subcategories always exist either properly or as null sets. Whether or not a given subcategory exists non-null depends on the restrictions demanded, the ground set, and the universe of discourse (as mentioned in footnote[1])).

The subcategory $\mathfrak{S}(G)$ of $\mathfrak{D}(G)$ consists of those elements of $\mathfrak{D}(G)$ whose distance functions satisfy the condition of

Symmetry: $\qquad d(x, y) = d(y, x) \qquad (\forall x, y).$

The subcategory $\mathfrak{V}(G)$ of $\mathfrak{D}(G)$ consists of those elements of $\mathfrak{D}(G)$ whose distance functions satisfy the condition of vanishing:

$$d(x, x) = d(y, y) \neq d(x, y) \qquad (\forall x, y \ni x \neq y).$$

Since G is an abstract set (in general) we agree to identify the elements $d(x, x)$ one of which is defined by each element of $\mathfrak{V}(G)$. The resulting element is labeled O and G is called a *zeroidal* ground set when one of its elements (and only one) is labeled O. We may then state the condition on the distance functions of the spaces in $\mathfrak{V}(G)$ as

Vanishing. $\qquad d(x, y) = 0$ if and only if $x = y$.

The set-theoretic cross cut $\mathfrak{S}(G) \cap \mathfrak{V}(G)$ is denoted by $\mathfrak{N}(G)$ and its elements are called *generalized semimetric spaces*. The name obviously derives from the fact that symmetry and vanishing are precisely the conditions imposed on those elements of $\mathfrak{D}(\mathfrak{R}^{*})$ which are called semimetric spaces.

Most of the distance spaces of interest are in $\mathfrak{N}(G)$. Examples to the contrary are

1. The directed ray of analytic geometry is in $\mathfrak{B}(\mathfrak{R})$ but not in $\mathfrak{S}(\mathfrak{R})$.
2. The complex (Euclidean) plane is in $\mathfrak{S}(\mathfrak{C})$ but not in $\mathfrak{B}(\mathfrak{C})$.

If S is a ground space in $\mathfrak{N}(G)$ whose distance function satisfies the condition of

Normality: $\qquad d(x, 0) = x \qquad (\forall x \in S)$

then S is called a *normal ground space* over G. The prototype of ground spaces; namely, $\mathfrak{R}^*$ bearing the Euclidean metric, is a normal ground space over $\mathfrak{R}^*$.

10. Restrictions on the ground set. A ground set is called *algebraic* if it forms a groupoid under a commutative operation $a+b$. If G is an algebraic ground set we may define for the elements of $\mathfrak{D}(G)$ the notions (which are of great importance in classical geometry and the geometry of metric spaces) of betweenness and convexity. Let $a, b, c \in S \in \mathfrak{D}(G)$. One says that b is *between* a and c, written (abc), provided $d(a, b)+d(b, c)=d(a, c)$. This is betweenness in the broad sense. Betweenness in the strict sense, written $(abc)^*$, demands that (abc) subsist and that a, b, c be pairwise distinct. Most of the literature dealing with betweenness in semimetric and metric spaces employs the strict notion. However, the broad notion of betweenness seems better adapted to more general distance spaces. It should be noted that (abc) and (cba) are equivalent propositions (as demanded by intuition) only because the groupoid on G has been assumed commutative. S is called *convex* if

$$a, c \in S \wedge a \neq c \cdot) \cdot \exists b \in S \ni (abc)^* \text{ subsists.}$$

S is called *externally convex to the right* provided

$$a, b \in S \wedge a \neq b \cdot) \cdot \exists c \in S \ni (abc)^* \text{ subsists.}$$

External convexity to the left is defined in an obvious similar fashion. If S is externally convex both to the right and to the left it is said to be *externally convex*.

If $S^* \subset S$, it is sometimes desirable to consider the set-theoretic product of all convex subsets (if any) of S which contain S^*. This product is also convex and is called the *convex closure* (convex cover, convex extension) of S^* and is written S^*_c. It does not, of course, exist for arbitrary S and S^*. Similar products may be defined to yield "closures" externally convex to either side or to both sides provided the appropriate subsets exist.

Suppose now that G is algebraic and is partially ordered (see **3**) by $\leq$. By definition $a \geq b \cdot\sim\cdot b \leq a$ and $a < b \cdot\sim\cdot a \leq a \wedge a \neq b$. We denote by $\mathfrak{T}(G)$ the subcategory of $\mathfrak{D}(G)$ whose elements have distance functions satisfying the

Triangle Inequality: $d(a,b)+d(b\ c)\geqq d(a,c)$.

The distance spaces of greatest interest in the classical case (d. h. $G=\mathfrak{R}^*$) are those in the set-theoretic product $\mathfrak{N}(G)\cap\mathfrak{T}(G)$ (assuming also that G is zeroidal as it is in the classical case). If 0 is the first element (0 denotes the 0 element of the vanishing condition) in the partial ordering of G by $\leq$ we say that any element of $\mathfrak{T}(G)\cap\mathfrak{N}(G)$ is a *generalized metric space* and we denote this product by $\mathfrak{M}(G)$.

If G is any partially ordered set then numerous "natural" topologies are defined in G each of which induces a topology (except for uniqueness of limits) in every element of $\mathfrak{D}(G)$ as we shall see shortly. This matter has been considered by FRÉCHET [12], [13], APPERT [1], [2], COLMEZ [5], [6], and DOSS [7], [8]. For the numerous topologies usually employed in partially ordered sets one may consult GARRETT BIRKHOFF's book [3].

11. Distance topology. Let G be a ground set in which a topology, $\lim_{n\to\infty} a_n$, is defined and let S be any element of $\mathfrak{D}(G)$. S in endowed with a topology (with the exception previously noted), called its *distance topology*, be the following definition: Let $a_1, a_2, \dots$ be a sequence of elements of S, then $\lim_{n\to\infty} a_n = a \in S \cdot\sim\cdot \lim_{n\to\infty} d(a,a_n)=0$ (assuming G zeroidal). We note once again that this distance topology may not be a topology, strictly speaking, in the sense of Section **3**, since this definition may fail to assign unique limits (as is the case in some classical semimetric spaces which are not metric). Additional restrictions both on the topology in G and on the distance function of S are necessary in order to assure unique limits in the distance topology of S. Additional restrictions of various types on the topology of G and the distance function of S obviously imply restrictions, in general, on the distance topology of S.

If $S, S^* \in \mathfrak{D}(G)$, where G is topologized and zeroidal, and if $S \approx S^*$, it is clear that S and S^* are homeomorphic in their respective distance topologies (assuming these are single-valued) where by a homeomorphism we mean any biuniform mapping of one onto the other which preserves accumulation elements in both directions. If S and S^* have the same point set and are homeomorphic in their respective distance topologies by the identity mapping, we say that the distance functions of S and S^* are *topologically equivalent*.

If $\mathfrak{D}(G)$ has a topologized, zeroidal ground set which is partially ordered and $S \in \mathfrak{D}(G)$ we may define completeness in the sense of FRÉCHET [4] for the distance topology in S (or for an arbitrary topology in S with G merely partially ordered and zeroidal) by defining CAUCHY sequences of S in the obvious way. Other topological notions which apply to elements of $\mathfrak{D}(G)$ when G is topologized and zeroidal were mentioned in Section **3.**

12. Path length. Let C be a chain (simply ordered set) and G arbitrary.

If $C^* \subset C$ and $f: C^* \to S^* \subset S \in \mathfrak{D}(G)$ is a single-valued mapping of C^* onto S^* then S^* is called the *path by f of C^* in S*. Suppose now that G is zeroidal, algebraic, and partially ordered. Denote by P any finite subset $p_1, \ldots, p_n$ of C^* written in its order in C and denote by $\{P\}$ the class of all such P. If S^* is the path by f of C^* in S, the *path length* of S^*, $L^f_{C^*}(S^*)$ is defined to be

$$\underset{\{P\}}{\text{l. u. b.}} \sum_{i=2}^{n} d(f(p_{i-1}), f(p_i)) \qquad \text{provided this l. u. b.}$$

exists in G. This definition may be extended, in the case where G is also a groupoid under a second commutative operation, in the obvious fashion to give an (upper) path integral of any single-valued function from a path in S to G. A lower path integral would be obtained by taking g. l. b. rather than l. u. b. and a theory of curvilinear integration may thus be developed in an extremely broad setting. To make this procedure more clear we list the proper definitions. Let G be as above and closed under a second binary single-valued operation ab. We shall assume (since it appears to be the case of most interest) that G is commutative under ab. Let C, S, C^*, and S^* and f be as above. Let g be a single-valued function on S^* to G. We define the upper and lower path integrals of g over $S^* = f(C^*)$ as

$$(u)\int_{S^*} g = \underset{\{P\}}{\text{l. u. b.}} \sum_{i=2}^{n} g(f(p_i))\, d_{i-1,i} \quad \text{and} \quad (l)\int_{S^*} g = \underset{P}{\text{g. l. b.}} \sum_{i=2}^{n} g(f(p_i))\, d_{i-1,i},$$

where $d_{i-1,i}$ in both definitions stands for $d(f(p_{i-1}), f(p_i))$.

One also sees how, using path integrals, one might define an integral of a function defined over an arbitrary subset of S as in the last paragraph by considering l. u. b. and g. l. b. of the path integrals for all possible paths in the arbitrary set. If G contains a unit element 1 for the operation ab (as in the last paragraph) one may define, as usual, characteristic function of a point of S and a theory of measure (measures being elements of G) might then be developed using the integral of the characteristic function of an arbitrary set over that set as the measure of the set. A theory of measure has not, to the best knowledge of the present writer, been developed in this particular fashion even for a general metric space where the measures would be real numbers, although E. J. McShane uses a Lebesgue integral defined without use of the concept of measure (other than that of an interval) to define measure as the integral of the characteristic function in Euclidean n-space in his book on integration.

13. Some special notions associated with ground spaces. If G is an algebra (as defined in Section **3**) and G forms a ground space in $\mathfrak{D}(G)$ under a distance function which is defined in terms of the operations already extant in G, we say that G is an *algebraically metrized ground space*. An obvious problem of interest in an algebraically metrized ground space is the study of relations between algebraic properties of the original operations in G and distance-theoretic properties of the distance function in G Examples of algebraically metrized ground spaces are the naturally oriented groups and autometrized Boolean algebras considered in **19.**

If G is any ground space, then G forms a groupoid under its distance function $d(x, y)$. We call this groupoid the *metroid* of G. It appears that there may be a particularly close association between the classical algebraic properties (commutativity, possession of a unit element, possession of inverses, nilpotency, etc.) of the metroid of G and distance-theoretic properties of G considered as a distance space. The present writer hopes to make a study of these relations in the near future.

Let G be a ground space in $\mathfrak{S}(G)$. G is said to *admit reflection in* $x \in G$ provided

$$y \in G \cdot) \cdot \exists | y' \in G_\varepsilon y \neq y' \wedge d(x, y) = d(x, y') \wedge d(y', z') = d(y, z);\ \forall y, z \in G \wedge y \neq x.$$

That is, there is a motion of G leaving only x fixed and such that all other elements are equidistant with their respective images from x. The latter requirement is evidently satisfied automatically whenever the first is, since, if a motion of G leaves x fixed, then y and its image must be equidistant from x. Thus, reflections may be regarded as motions which leave *exactly one* element fixed. The subject of motions which leave one or more points of a distance space fixed may well prove of interest as it has done in the case of topology where many studies have been made concerning homeomorphisms of topological spaces into or onto themselves which leave certain subsets fixed.

One of the spaces which we study later (see **19**) is the autometrized Boolean algebra. This is a ground space and its distance function possesses a very unusual feature which we call the *property of triangular fixity.* A ground space is said to have the property of triangular fixity provided $d(a, b) = c \cdot) \cdot d(a, c) = b \wedge d(b, c) = a$. That is, in words, two arbitrary points of the space and the third point which is the length of the original "segment" form a triangle each vertex of which is the length of the opposite side.

14. Other notions of distance-theoretic importance. Let $S \in \mathfrak{D}(G)$. A set $S^* \subset S$ is said to be a *metric base* for S provided that if f is a function on S^* to G there is at most one element, w, of S for which $d(w, s) = f(s)$; $\forall s \in S^*$. That is, in words, there is at most one element of S having distances from the elements of S^* agreeing with a preassigned set of elements of G. If there is *exactly one* such element for each set of preassigned elements of S, we say that S^* is a complete metric base for S. In the latter case, one sees that the distances of a point of S from the points of S^* (taken in some fixed order) constitute a set of coordinates in S which may be referred to (as in the classical case) as metric coordinates in S. Metric bases have proved of much interest in classical spaces and in metric spaces. We mention here an interesting conjecture, unproved at the present, due to Mr. JERRY GADDUM to the effect that in a metric space any equilateral set containing the maximal number of elements (that is, any maximal equilateral set) forms a metric base for the space provided the space is complete and convex. We remark, however, that is seems probable that external convexity is also required in addition to completeness and convexity to make this conjecture valid.

Let $S \in \mathfrak{D}(G)$. S is said to be *irreducible over G* (*irreducible over the ground set*) if every element of G is taken on as a distance in S. That is, S is irreducible over G if $G = D(S)$. In the classical case, the present writer has conjectured that every compact metric space which is irreducible over R^* (the non-negative real ray) contains a non-degenerate continuum (compact, connected set of more than two distinct elements). This conjecture is also undecided, but a simple example may be constructed to show that the conjecture is not valid if the restriction of compact be dropped.

Let $S \in \mathfrak{D}(G)$ and $D(S)$ be, as usual, the distance set of S. S is said to be *distancially irreducible* provided there is no *proper* subset S^* of S with $D(S^*) = D(S)$. That is, in words, S is distancially irreducible provided the deletion of any point of S diminishes the distance set of S. This property would appear to be of interest in connection with the formal study of distance sets. That is, the problem of realization of a distance set (given a distance set does there exist a space of a required type of which the given set is the distance set) has, in general, multiple valued solutions each solution being distancial to each other solution. One naturally inquires then as to the possibility of a *minimal* realization of a given distance set, or, equivalently, a *minimal* representative of a distanciallity class, minimal being in the sense of distancially irreducible.

A definition with applications to the study of the subset problem is that of *pseudo-S-set.* Let $S \in \mathfrak{D}(G)$ and $\mathfrak{U}(G)$ be a subcategory of $\mathfrak{D}(G)$ in which S has best congruence order $n > 1$. An element $S^* \in \mathfrak{U}(G)$ is said to be a pseudo-S-set if each $n-1$ points of S^* are imbeddable (congruently imbeddable) in S but $S^* \overline{\subseteq} S$.

The congruence indices (see **8**) of a space in a category of spaces over the same ground set are considered to be ordered lexicographically. Thus the *best congruence symbol* (a notion due to L. M. BLUMENTHAL) is obtained by minimizing first the first number and secondly the second number in a set of congruence indices. It should be noted that the first number of the best congruence symbol is not necessarily the best congruence order even when both are defined.

Let $S \in \mathfrak{D}(G)$ and S' and S'' be two subsets of S. S'' is said to be *coverable* (*congruently coverable*) in S by S' if there is a motion of S sending

S' into S'' so that $S^* \subset S''$. It should be noted here that, in general, $S^* \Subset S'$ does not imply S^* coverable by S' in S although the converse proposition is always valid. If, however, S has the property of free mobility, then $S^* \Subset S'$ does imply S^* coverable in S by S'.

Let $S \in \mathfrak{D}(G)$. S is said to be *metrically irreducible* provided there is no *proper* subset $S^* \subset S$ with $S^* \approx S$ The property of being metrically irreducible is not universal even in the classical cases. For example, the real Hilbert space is congruent to one of its proper subsets. LINDENBAUM has proved, however, (see reference given in [4]), that no subset of a compact metric space which is both an F_σ and a G_δ [4] can be congruent to a subset of itself. LINDENBAUM uses the term monomorphic rather than metrically irreducible.

Let $S, S^* \in \mathfrak{D}(G)$. S and S^* are said to be *metrically compatible* provided $S \Subset S^* \wedge S^* \Subset S \cdot) \cdot S \approx S^*$. Obviously, two distance spaces with the same ground set are metrically compatible provided one of them is metrically irreducible. A category in $\mathfrak{D}(G)$ is called metrically compatible if each two of its elements are metrically compatible. Thus a subcategory of $\mathfrak{D}(G)$ is metrically compatible if all save possibly one of its elements are metrically irreducible. The notion of compatibility is also of interest in the topological case (where homeomorphism replaces congruence). For example, an unsolved problem is whether or not FRÉCHET's space E_ω is homeomorphic to the real Hilbert space although it is known that each is topologically imbeddable in the other.

Finally, we mention in this section the metrization problem. Suppose that $\mathfrak{D}(G)$ has a topologized ground set. Let S be a topological space (either a Fréchet limit class or some other variety such as a Hausdorff space). The metrization problem consists in giving necessary and sufficient conditions that a topological space of a given type be homeomorphic to an element of $\mathfrak{D}(G)$ bearing its distance topology (assumed again to yield unique limits). Of course, these conditions are to be stated in terms of concepts definable from the topology of the given type of topological space. The metrization problem is a special case of a wide class of problems which inquire what topological properties imply certain distance-theoretic properties up to homeomorphism. An example of one of these problems, other than the metrization problem itself, in the classical case is KARL MENGER's convexification problem. Namely, what topological and metric properties must a metric space possess (other than convexity itself) in order that it be homeomorphic to a convex metric space?[5]) No attack has been made on the metrization problem in the

[5]) A solution of the metrization problem giving necessary and sufficient conditions (previously unobtained) in order that a regular Hausdorff space (not necessarily satisfying a countability axiom) be metrizable has been obtained by R. H. BING, (*Bull. Amer. Math. Soc.*, **56** (1950), p. 53 contains an abstract of this result), E. E. MOISE and R. H. BING, working independently, have recently solved the convexification problem.

case of ground sets which are merely topologized, but a study has been made of the special case where the ground set is a particular type of group [15].

The author has tried in the preceding part of this paper to give merely an indication of a few of the classical notions which have meaning in abstract distance spaces (with restricted ground sets in general). There are, of course, others too numerous to mention here. It is also to be expected that the study of abstract distance spaces rather than more familiar types will yield many notions heretofore unconsidered because of their failure to exist in the more familiar spaces. In fact, this is already the case for triangular fixity as defined above. Although this notion is definable in any ground space and is a very simple and obvious possibility, it was not previously considered because none of the spaces under study had the property of triangular fixity. It would possibly be of interest to enquire as to what familiar conditions, if any, are placed on a metric space whose point set is the non-negative real ray by the property of triangular fixity.

15. A preface to the results. In the following part of this paper we give a survey of the results, other than in the classical case, which have been obtained up to the present in studies concerned with the geometry of abstract distance spaces. This section contains a few preliminary remarks to this survey.

Several studies have been made of distance spaces in which the primary aim is considerations of a purely topological rather than distance-theoretic nature. The ground sets in these studies have been ordered groupoids, ordered groups, chains, lattices, ordered fields, and classes of probability functions. The last of these (**19**) begins a metric approach to the foundations of chunk topology (topology without points). These studies are certainly of much interest and may prove of aid in distance-theoretic studies concerned with path integrals and related topics requiring ordering or topological considerations. Since our major interest in this paper is, however, the study of distance geometry proper we shall content ourselves with merely giving references to the above mentioned papers. These are: G. B. Price [22], O. H. Hyers [14], G. K. Kalisch [15], A. Appert [1], [2], R. Doss [7], [8], Maurice Fréchet [12], [13], Jean Colmez [5], [6], W. Krull [17], and Karl Menger [19].

Studies concerned with distance geometry proper (and closely related topics) have involved four types of ground sets. These are groups, Abelian groups, fields, and Boolean algebras. In the case of arbitrary groups and Boolean algebras, most of the work has been concerned with a particular type of ground space. In the case of Abelian groups, most of the work is on a space which is a ground space over the set of elements of the group paired with their negatives. In the case of fields, most of the work concernes imbeddability of distance spaces over the field in vector spaces metrized in a more or less Euclidean fashion over the field. Thus it is easily seen that work has only begun even for these special types of ground sets. It seems probable that almost all of the classical theorems relating to metric spaces can be carried over to the category $\mathfrak{M}(F)$ where F is an ordered field and a study of analogous theorems for the category $\mathfrak{M}(B)$ where B is a Boolean algebra endowed with one of its natural topologies may well prove of equal interest. We leave to the reader the suggestion of other programs of research which are corollary to the abstraction of the notion of distance and distance space.

16. Naturally oriented groups. Let G be an additively written group. The point set of G is made a ground space in $\mathfrak{B}(G)$ by the definition $d(a, b) = b - a$. Following MENGER we call such spaces *naturally oriented groups*. The naturally oriented group is obviously a direct generalization to arbitrary groups of the directed line of analytic geometry which is a naturally oriented group on the additive group of reals. In G, $d(a, b) = -d(b, a)$ and we denote by $\mathfrak{A}(G)$ the subcategory of $\mathfrak{B}(G)$ whose distance functions have this property of antisymmetry or skewness. The elements of $\mathfrak{A}(G)$ are called (again after MENGER) G-oriented spaces. Naturally oriented groups and G-oriented spaces have been studied by KARL MENGER [18], OLGA TAUSSKY [24] and KESTELMAN and SMITH [16].

The principal result obtained by MENGER is a solution of the subset problem for G in $\mathfrak{A}(G)$ as follows:

Theorem 16. 1. (MENGER) *Let* $M \in \mathfrak{A}(G)$. $M \subseteq G \cdot \sim \cdot p, q, r \in M \cdot) \cdot d, p, q + d(r, p) + d(q, r) = 0$.

As a corollary we have:

Corollary 16. 1. 1. *The naturally oriented group G has congruence order three in* $\mathfrak{A}(G)$.

The principal result of KESTELMAN and SMITH (for our viewpoint) is a theorem concerning distance sets in a certain type of set-theoretic decomposition of G. This theorem we interpret as follows:

Theorem 16. 2. (KESTELMAN and SMITH) *Let G_1 be a subgroup of G and suppose that G_1 has a subgroup G_2 of index k (k any cardinal) in G_1. Then G is the union of k disjoint sets, pairwise superposable by (group) translation, the (set-theoretic) sum of whose distance sets does not intersect $G_1 - G_2$ (set-theoretic difference).*

The result of TAUSSKY which has major significance for us is:

Theorem 16. 3. (TAUSSKY) *The (group) reflections of G are motions of G if and only if G is either an Abelian group or a Hamiltonian 2-group* [6])

[6]) A Hamiltonian 2-group is a group in which each non-zero element has period which is a power of 2.

Theorem 16. 3 makes it evident that the superposability properties and group of motions of a naturally oriented group differ considerably, in general, from those of a naturally metrized group as considered in **17.** below.

Olga Taussky has also considered similarity geometry for such groups as naturally oriented groups and others with distances closely related.[7])

17. Naturally metrized groups. Let G be an additively written abelian group and $|G|$ the set of all *unordered* pairs $(a, -a)$ for $a \in G$. We write $|a|$ for $(a, -a)$ and identify $|0|$ with 0. The elements of $\mathfrak{N}(|G|)$ are called *G-metrized spaces.* The point set of G is made a G-metrized space by the definition $d(a, b) = |a-b|$ for $a, b \in G$. G bearing this distance function is called a *naturally metrized group* and naturally metrized groups are obviously a direct generalization to arbitrary abelian groups of the Euclidean line which is a naturally metrized group on the additive group of reals. (Provided we agree to identify $a, -a$ with ordinary absolute value.)

Naturally metrized groups have been studied by Karl Menger [18], [21], Olga Taussky [24], and the writer [11][8]), and many results have been obtained for these spaces.

We first make a few general remarks about distances in G which were originally observed by Menger. We say that $|a|$ has the same order (period) as does a. We denote by G_2 the subgroup of G consisting of all elements of G whose orders do not exceed 2, by $R_a(b)$ (called the reflection of b in a) the set

$$R_a(b) = \{x \in G \mid d(a, x) = d(a, b)\},$$

and by $S(a, b)$ (called the symmetral of a and b) the set

$$S(a, b) = \{x \in G \mid d(a, x) = d(b, x)\}.$$

We denote by $o(a)$ and $o(|a|)$, respectively, the orders of a and $|a|$.

Remark. $o(d) > 2 \wedge a \in G \cdot) \cdot \exists | b, c \in G \ni d(a, b) = d(a, c) = |d|.$

$$o(d) = 2 \wedge a \in G \cdot) \cdot \exists | b \in G \ni d(a, b) = |d|.$$

Remark. $a, b \in G \wedge o(d(a, b)) > 2 \wedge d\ e \in G \cdot) \cdot \exists | c \in G \ni d(a, c) = |d| \wedge d(b, c) = = |e|.\ a, b \in G \wedge o(d(a, b)) = 2 \wedge c \in G \wedge \min\ (o(d(a, c)), o(d(b, c))) >$

$$> 2 \cdot) \cdot \exists | c' \in G \ni c \neq c' \wedge d(a, c) = d(a, c') \wedge d(b, c) = d(b, c').$$

Remark. *$R_a(b)$ contains at most one element; $\forall a, b \in G$. $a, b, c \in G \wedge R_a(c)$ non-null $\wedge R_b(c)$ non-null $\cdot) \cdot d(R_a(c), R_b(c)) = |2(a-b)|$.*

Theorem. 17. 1. (Menger) *$a, b \in G \wedge a \neq b \cdot) \cdot S(a, b)$ null $\vee S(a, b) \approx G_2$. If $G = G_2$ and $a \neq b$ then $S(a, b)$ is null. If G is a cyclic group of odd order then $S(a, b) \approx G_2$ for $a \neq b$.*

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7) Olga Taussky, Über ähnliche Abbildungen von Gruppen, *Ergebnisse eines Math. Koll. (Wien),* **3** (1931), 13—14.

8) The author wishes to take this opportunity to make the following correction of a typographical error in this paper [11]: On page 639 in the 5th paragraph, third line, "for all $a, b \in G$" read "for all $a, b \in S$".

We now examine the superposability properties of G. They have been studied by MENGER and the writer. The results are:

Theorem 17. 2. (MENGER) *If $a, a' \in G$, $a \neq a'$ there are exactly two motions of G sending a into a' (namely, $x' = a' + x - a$ and $x' = a' - x + a$) unless $G = G_2$ in which case these two motions coincide.*

Theorem 17. 3. (MENGER) *G has the property of two point superposability. If $a, b \approx a', b'$ and $a \neq b \neq b'$ the motion superposing a, b and a', b' is unique unless $o(d(a, b)) = 2$ and $G = G_2$ in which case there are exactly two such motions.*

Finally we list the property of free mobility of G and give the structure of the group of motions of G in terms of the structure of G.

Theorem 17. 4. *G has the property of free mobility. $\mathfrak{G}(G)$ may be obtained as the extension of the group of (group) translations $(x' = x + a)$ by the reflection in the origin $(x' = -x)$.*

Theorem 17.4 was suggested, of course, by Theorems 17.2 and 17.3 and they may be obtained as corollaries to 17.4.

Characterization problems for naturally metrized groups have been studied by MENGER, TAUSSKY, and the writer. (The theorem of the author stated below is previously unpublished).

Remark. (MENGER) *Any G-metric space (G-metrized space) containing not more than two elements is imbeddable (congruently imbeddable) in G.*

Theorem 17. 5. (MENGER) *Let $p_1, p_2, p_3 \in S \in \mathfrak{N}(|G|)$. The set $p_1, p_2, p_3 \subseteq G \cdot \sim \cdot e_i = \pm 1$; $i = 1, 2, 3$; may be chosen so that $e_1 r_{23} + e_2 r_{31} + e_3 r_{12} = 0$ where $|r_{ij}| = d(p_i, p_j)$; $i, j = 1, 2, 3$.*

Corollary 17. 5. 1. *If G contains an equilateral triple the "edge length" is order three.*

Theorem 17. 6. (MENGER) *The congruence order of G in the subcategory of $\mathfrak{N}(|G|)$ whose elements contain exactly four points and have at least one distance of order 2 is 3.*

Corollary 17. 6. 1. *The distance set of a pseudo-G-quadruple constains no elements of order 2.*

Theorem 17. 7. (MENGER) *The best congruence order of G in $\mathfrak{N}(|G|)$ is four.*

The preceding theorem may be sharpened by abandoning the notion of congruence order as in

Theorem 17. 8. *Let $S \in \mathfrak{N}(|G|)$ and $a, b \in S$, $a \neq b$. If each quadruple of points of S which contains a and b is imbeddable in G then $S \subseteq G$.*

Theorem 17. 9. (MENGER) *If p_1, p_2, p_3, p_4 form a pseudo-G-quadruple in $\mathfrak{R}(|G|)$ then $2r_{12}=2r_{34}$, $2r_{13}=2r_{24}$, and $2r_{14}=2r_{23}$ where $|r_{ij}|=d(p_i, p_j)$; $i, j=1, 2, 3, 4$.*

Theorem 17. 10. (MENGER) *If $p_1, p_2, p_3, p_4 \in G$ and n is a natural integer so that $nr_{12}=nr_{34}$, $nr_{13}=nr_{24}$, and $nr_{14}=nr_{23}$, where $|r_{ij}|=d(p_i, p_j)$; $i, j=1, 2, 3, 4$; then the distance set of this quadruple (p_1, p_2, p_3, p_4) contains an element of order n.*

Theorem 17. 11. (MENGER) *If G contains no elements of order 2 and p_1, p_2, p_3, p_4 are four points of an element of $\mathfrak{R}(|G|)$ and if each three points of this quadruple are imbeddable in G, then the four points form a pseudo-G-quadruple if and only if $d(p_1, p_2)=d(p_3, p_4)$, $d(p_1, p_3)=d(p_2, p_4)$, and $d(p_1, p_4)=d(p_2, p_3)$.*

Remark. (TAUSSKY) *If each three points of a G-metric quadruple, p_1, p_2, p_3, p_4, are imbeddable in G then a necessary (but not, in general, sufficient) condition that the four points form a pseudo-G-quadruple is that $2r_{12}=$ $=2r_{34}$, $2r_{13}=2r_{24}$, and $2r_{14}=2r_{23}$, where $|r_{ij}|=d(p_i\ p_j)$ $i, j=1, 2, 3, 4$.*

Remark. (TAUSSKY) *If each three points of a G-metric quadruple, p_1, p_2, p_3, p_4, are imbeddable in G then a sufficient (but not, in general, necessary) condition that the four points form a pseudo-G-quadruple is that $d(p_1, p_2)=$ $d(p_3, p_4)$, $d(p_1, p_3)=d(p_2, p_4)$, and $d(p_1, p_4)=d(p_2, p_3)$.*

From the two preceding remarks and Theorems 17.9—.17.11, it is seen that the structure (distance-theoretic structure) of pseudo-G-quadruples is very similar to that of pseudolinear-quadruples [20] as one would expect. The following theorem, due to OLGA TAUSSKY, completely characterizes pseudo-G-quadruples among G-metric quadruples (in $\mathfrak{R}(|G|)$).

Theorem 17. 12. (TAUSSKY) *Let p_1, p_2, p_3, p_4 be a G-metric quadruple (in $\mathfrak{R}(|G|)$) with $d(p_i, p_j)=|r_{ij}|$; $i, j=1, 2, 3, 4$. A necessary and sufficient condition that the quadruple be a pseudo-G-quadruple is that $2r_{ij}\neq 0$; $i, j=$ $=1, 2, 3, 4$, $i\neq j$; and it be possible to choose $e_{ij}=\pm 1$; $i, j=1, 2, 3, 4$, $i\neq j$; so that*

$$\begin{aligned}
e_{12}r_{12}+e_{23}r_{23}+e_{13}r_{13}&=0\\
e_{12}r_{12}+e_{24}r_{24}+e_{14}r_{14}&=0\\
e_{13}r_{13}+e_{14}r_{14}+e_{34}r_{34}&=0\\
e_{23}r_{23}+e_{24}r_{24}+e_{34}r_{34}&=0.
\end{aligned}$$

Theorem 17. 13. (MENGER) *If G contains no elements of order 2 then the quasi-congruence order of G in the subcategory of $\mathfrak{R}(|G|)$ whose elements contain no equilateral triples is 3.*

TAUSSKY has shown that the restriction of Theorem 17. 13 that G contains no elements of order 2 is unnecessary. Thus

Theorem 17.14. (TAUSSKY) *The quasi-congruence order of G in the subcategory of $\mathfrak{N}(|G|)$ whose elements contain no equilateral triples is* 3.

MENGER [21] has considered a generalization of path length in arbitrary (not necessarily partially ordered) naturally metrized groups. It differs from our generalization in **12** in that it assigns as path length not an element but, in general, a subset of G. For $|a_1|, |a_2|, \ldots, |a_n| \in |G|$ we define $\sum_{i=1}^{n} |a_i|$ to be the set of 2^n elements $\sum_{i=1}^{n} e_i a_i$ obtained by assigning to the e_i the values ± 1 in all possible fashions. If $p_1, p_2, \ldots, p_n$ is a chain (ordered set) of elements of a G-metric set we assign as path length $L(p_1, \ldots, p_n) = \sum_{i=1}^{n-1} d(p_i, p_{i+1})$. For E any (finite) chain in any G-metric space it is immediate that $L(E)$ is its own reflection in 0 and is a congruence invariant. If F is a subset of E with the points of F ordered as in E and F contains the endpoints of E then $L(F) \subset L(E)$.

Suppose now that G is a topological group (G bears a topology in which $a-b$ is continuous and the underlying topological space is a FRÉCHET limit class). It is also assumed that derived sets are closed in G. An arc in G is any homeomorph of the unit interval $[0, 1]$. If the topology of G is the metric topology of a metric space (that is, if G is a metric group) the usual arc length, $l(A)$, is attached to any arc A of G. (For a definition of $l(A)$ see [4]).

If B is an arc in G we define $L^*(B)$ as the union of all $L(E)$ where E is a finite subset of B carrying the order induced in B by the given homeomorphism with $[0, 1]$. (It can be easily seen that this order is independent of the homeomorphism chosen except possibly for a total inversion of order by an interchange of the endpoints). We also require that all of the sets E contain the endpoints of B. The *length set* (Längenmenge), $L(B)$, of the arc B is then defined as the closure (in G) of $L^*(B)$.

The results, all due to KARL MENGER, are:

Theorem 17.15. *If E_k is a sequence of ordered n-tuples $p_1^k, p_2^k, \ldots, p_n^k$ of points of G and E is an n-tuple $p_1, p_2, \ldots, p_n$ of points of G with $\lim_{k\to\infty} p_i^k = p_i$; $i = 1, 2, \ldots, n$; then $\lim_{k\to\infty} L(E_k) = L(\mathrm{E})$.*[9] *Path length of ordered k-tuples is a set-valued function which is continuous in the coordinate convergence of ordered k-tuples.*

[9]) Limit here is used in the sense of set theory. This means that $L(E)$ has the property any neighborhood of one of its points intersects all but a finite number of the $L(E_k)$ and that any point whose every neighborhood intersects an infinite number of the $L(E_k)$ is contained in $L(E)$.

Remark. *If B and B^* are arcs in G with $B \approx B^*$ then $L(B) = L(B^*)$.*

Theorem 17. 16. *$L(B)$ is an upper semicontinuous function of arcs in G in the set-theoretic topology of subsets of G (see footnote[9]), here, of course only lower limits are under consideration). More precisely, $\lim\limits_{n \to \infty} B_n = B$ implies $L(B) \subset \lim\limits_{n \to \infty} L(B_n)$ where the first limit is in the sense of coordinate convergence (the coordinates are given by the homeomorphisms defining the arcs) and the second limit is in the sense of lower topological limit of a sequence of sets.*

Theorem 17. 17. *If F is a subset of an arc B and is dense in B then $L(B)$ is the union (more precisely, the closure of this sum) of all $L(E)$ where E is a finite subset of F.*

Theorem 17. 18. *Let $\{F_n\}$ be a sequence of finite subsets of an arc $B \subset G$, each carrving the order on B, each of which contains the endpoints of B, with $F_n \subset F_{n+1}$ so that the set-theoretic sum $\bigcup\limits_{n=1}^{\infty} F_n$ is dense in B. Then $L(B)$ is the closure of $\bigcup\limits_{n=1}^{\infty} L(E_n)$.*

MENGER does not supply proofs of the foregoing theorems on length sets in his paper [21]. The author has found, however, that they may be verified in a straightforward manner.

Remark. *An interesting unsolved problem is as follows: If B is an arc in G is it necessary that 0 $L(B)$? An affirmative answer implies that $L(B^*) \subset L(B)$ provided $B^* \subset B$.*

Remark. *Another interesting unsolved problem is as follows: Let G be a metric group and B, B^* arcs in G. Under what conditions does $L(B) = L(B^*)$ imply $l(B) = l(B^*)$ and under what conditions does the converse proposition hold? This question was examined by* MENGER *for the group of vectors of the Euclidean plane.* [21].

Remark. *It seems probable that the foregoing theorems on length should be of particular interest when interpreted in the special case of the group of vectors of Euclidean n-space. This possibility is apparently unexploited.*

In **17**, we remark that the present author has recently made a study of a modified type of distance set in naturally metrized groups. An abstract of this study is to be found in the *Proceedings of the International Congress of Mathematicians, Cambridge,* 1950.

18. Square-metrized vector spaces. OLGA TAUSSKY [23] has considered the following type of distance spaces: Let P be a field of characteristic zero (non-modular field) and let P_n be the n-dimensional vector space over P whose elements are ordered n-tuples of elements of P. Define distance in

P_n by $d(p,q)=\sum_{i=1}^{n}(p_i-q_i)^2$. The squares of the distances of elements in the naturally metrized additive group of P are then the distances of the elements in P_1. We call P_n the *n-dimensional square-metrized vector space over P.* We consider the category $\mathfrak{B}(P)$ of $S(P)$ whose elements satisfy the one way vanishing condition: $p=q$ implies $d(p,q)=0$. The elements of $\mathfrak{B}(P)$ we call *P-metrized spaces.* A subset K of P_n is linearly dependent if there are $n+1$ elements $e_1,\dots,e_{n+1}$ of P, not all 0, so that for each $x\in K$, $\sum_{i=1}^{n} e_i x_i + e_{n+1}=0$. P is called a *Pythagorean field* if each sum of squares in P is itself a square in P. P is called a *square field* if each element of P is a square in P. For $p_0,\dots,p_k$ elements of a P-metrized space we denote by $D(p_0,p_1,\dots,p_k)$ the determinant of order $k+1$ whose element in the ith row and jth column is $d(p_i,p_j)$ bordered by a row and column of units intersecting in 0. (We assume the existence of 1 in P unless a definition of field is used which includes this.)

The results, all due to Olga Taussky, are:

Theorem 18. 1. *In order that each finite P-metrized space be imbeddable (congruently imbeddable) in P_n for some n it is necessary and sufficient that P be not formally real (that is, -1 is a sum of squares in P). If -1 is the sum of u squares in P then every P-metrized space of n points is imbeddable in $P_{s(n)}$ where $s(n)=(n-1)u+\binom{n}{2}$.*

Theorem 18. 2. *If P is a Pythagorean field each linearly dependent subset of P_n is congruently contained in P_{n-1} for $n>1$.*

Theorem 18. 3. *If P is a square field then*

1. *Every P-metrized $(n+1)$-tuple is imbeddable in P_n.*

2. *In order that a P-metrized $(n+2)$-tuple be imbeddable in P_n it is necessary and sufficient that $D(p_0,\dots,p_{n+1})=0$.*

3. *A P-metrized $(n+3)$-tuple is imbeddable in P_n if and only if each of its $(n+2)$-tuples is so imbeddable and $D(p_0,\dots,p_{n+2})=0$.*

4. *P_n has best congruence order $n+3$ in $\mathfrak{B}(P)$.*

Remark. *It is clear that the definition of distance in P_n is analogous to the square of the distance in the (real or complex) Euclidean n-dimensional space. In fact, if P is a square field we may actually make a direct generalization of Euclidean n-space to P_n. Theorem* 18. 3 *above is exactly analogous to the corresponding results for the Euclidean n-space when $\mathfrak{B}(P)$ is replaced by the more restrictive $\mathfrak{R}(P)$* [20]. *It seems probable, as remarked in* **15**, *that if P is ordered (or, for some purposes, merely topologized) a great many of the classical results go through in these P-metrized spaces.*

19. Autometrized Boolean algebras. The writer has considered spaces (in particular, a certain type of ground space) over Boolean algebras [9], [10].

We proceed as follows: Let B be a Boolean algebra (see **3**) with meet, join, complement, and inclusion (in the wide sense) denoted by $a \wedge b$, $a \vee b$, a', and $a \subset B$, respectively. Define distance in B by $d(a, b) = (a \wedge b') \vee (a' \wedge b)$ (the so called "symmetric difference"). We call B then an ***autometrized Boolean algebra*** and it seen to be a normal ground space in $\mathfrak{N}(B)$. It should be noted that this definition of distance function is exactly the composition of the additive group of the Boolean ring associated with B,[10]) and hence results concerning distances in B may also be considered as results on the additive group of a Boolean ring with unity. This yields a new link between one of the more familiar types of algebras and an abstract distance space. The results are:

Theorem 19. 1. *B is a normal ground space in* $\mathfrak{M}(B)$.

Theorem 19. 2. *Complementation yields a motion of B. No non-trivial "translation" (meet by a fixed element other than I or join by a fixed element other than* 0[11]) *is a motion of B.*

Theorem 19. 3. *Betweenness in B is equivalent to the lattice betweenness of* PITCHER *and* SMILEY[12]).

Theorem 19. 4. *If B is a normed (metric) lattice* [3] *then betweennes as defined by the distance function in B is equivalent to betweenness as defined by the (real) metric of B.*

Theorem 19. 5. *B has the property of free mobility.*

Theorem 19. 6. *B has the property of triangular fixity.*

Theorem 19. 7. *Each point of B forms a complete metric base for B*

Theorem 19. 8. *If f is any motion of B then* $f(x') = f'(x)$; $\forall x \in B$. *In words: any motion is an automorphism for complementation.*

Theorem 19. 9. *The group of motions of B and the group of (Boolean algebra) automorphism of B have only the identity mapping in common*

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[10]) A Boolean ring (with unity) is a ring with unity in which multiplication is idempotent (see **3**). M. H. STONE (Subsumption of Boolean algebra under the theory of rings, *Proc. Nat. Acad. Sci.*, **20** (1934), 197—202) has shown that a Boolean ring with unity is obtained from a Boolean algebra by defining $a + b = (a \wedge b') \vee (a' \wedge b)$ and $ab = a \wedge b$, and, conversely, any Boolean ring with unity may be obtained in this fashion from some Boolean algebra.

[11]) 0 and I denote the first and last elements, respectively, of B.

[12]) PITCHER and SMILEY (Transitivities of betweenness, *Trans. Amer. Math. Soc.*, **52** (1942), 95—114) have defined a concept of betweenness in arbitrary lattices by the condition of GLIVENKO (Geometrie des systems de choses normees, *Amer. J. of Math.*, **58** (1036), 799–828):

(G) $$(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (b \vee c)$$

which is equivalent to metric betweenness (in the wide sense) in normed (metric) lattices [3].

although they are both subgroups of the group of automorphisms for complementation.

Remark. *An interesting conjecture of the writer is that the group of automorphisms for complementation of B. may be expressed as the (group) direct product of the group of motions of B and the group of (Boolean algebra) automorphisms of B.*

Theorem 19. 10. *Every motion of B is determined by the element into which it sends* 0. *The transformation is* $f(x) = d(x, f(0))$; $\forall x \in B$.

Theorem 19. 11. *The group of motions of B is isomorphic to the additive group of the Boolean ring associated with B. Hence, all the motions of B, other than the identity, are periodic of period* 2 *and the group of motions of B is abelian.*

Finally, we mention in connection with the subset problem for B in $\mathfrak{N}(B)$:

Theorem 19. 12. *B has best congruence order theree in* $\mathfrak{N}(B)$.

References.

[1] A. Appert, Écart partiellement ordonné et uniformité, *Comptes Rendus* (Paris), **224** (1947), 442-444.

[2] A. Appert, Espaces uniformes généralisés, *Comptes Rendus* (Paris), **222** (1946), 986-988.

[3] Garrett Birkhoff, Lattice theory, *Amer. Math. Soc. Colloquium Publications,* XXV. (Revised Edition, 1948.)

[4] L. M. Blumenthal, Distance geometries, *University of Missouri Studies,* XIII. (1938).

[5] Jean Colmez, Des espaces a écart, *La Revue Sci.,* **85** (1947), 39—41.

[6] Jean Colmez, Sur divers problèmes concernant les espaces topologiques, *Portugaliae Mathematica* **6** (1947), 119—244.

[7] R. Doss, Sur la condition de régularité pour l'écart abstrait, *Comptes Rendus* (Paris), **223** (1946), 14—16.

[8] R. Doss, Sur les espaces où la topologie peut être définie à l'aide d'un écart abstrait, symétrique et régulier, *Comptes Rendus* (Paris), **223** (1946), 1087—1088.

[9] David Ellis, Autometrized Boolean algebras, *Canadian J. of Math.* (To appear.)

[10] David Ellis, Autometrized Boolean algebras. II. — The group of motions of B. *Canadian J. of Math.* (To appear.)

[11] David Ellis, Superposability properties of naturally metrized groups, *Bull. Amer. Math. Soc.,* **55** (1949), 639-640.

[12] Maurice Fréchet, De l'écart numérique à l'écart abstrait, *Portugaliae Mathematica,* **5** (1946), 121-131.

[13] Maurice Fréchet, Sur les espaces à écart régulier et symétrique, *Boletin Soc. Portuguesa de Math.,* **1** (1947), 25—28.

[14] D. H. Hyers, Pseudo-normed linear spaces and abelian groups, *Duke Math. J.* **5** (1939), 628-634.

[15] G. K. Kalisch, On uniform spaces and topological algebra, *Bull. Amer. Math. Soc.,* **52** (1946), 936-939.

[16] H. Kestelman and C. A. B. Smith, On the distances between the elements of a subset of a group, *J. London Math. Soc.,* **24** (1949), 131—136.

[17] W. KRULL, Allgemeine Bewertungstheorie, *J. f. d. reine u. angew. Math*, **167** (1931) 160–196.

[18] KARL MENGER, Beiträge zur Gruppentheorie. I., *Math. Zeitschrift*, **33** (1931), 396—418.

[19] KARL MENGER, Statistical metrics, *Proc. Nat. Acad. Sci.*, **28** (1942), 535—537.

[20] KARL MENGER, Untersuchungen über allgemeine Metrik, *Math. Ann.*, **100** (1928) 75—163.

[21] KARL MENGER, Zur Begründung einer Theorie der Bogenlänge in Gruppen, *Ergebnisse eines Math. Kolloquiums* (Wien), **5** (1933), 1—6.

[22] G. BALEY PRICE, A generalization of a metric space with application to spaces whose elements are sets, *Amer. J. Math.*, **63** (1941), 546—560.

[23] OLGA TAUSSKY, Abstrakte Körper und Metrik. I., *Ergebnisse eines Math. Kolloquiums* (Wien), **6** (1933), 20—23.

[24] OLGA TAUSSKY, Zur Metrik der Gruppen, *Anzeiger der Akad. der Wissenschaften* (Wien, 1930), 140—142. — See also *Math. Zeitschrift*, **33** (1931), 412, and *Quart. J. Math.* (Oxford Ser.) **12** (1941), 64–67.

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