On approximative solution of algebraic equations.

By PAUL TURÁN in Budapest.

1. Many theoretical and technical questions lead to the problem to solve an algebraic equation with numerically given coefficients. To find approximative values for the zeros a considerable literature developed starting with NEWTON. Most of these deal exclusively with equations with real coefficients and — what is more important in some problems of mechanics and electricity — even in this case they furnish approximative values for the real zeros only. The fundamental idea of the only known method which can be applied to the approximation of real zeros as well as of the complex ones appears in the papers of NEWTON, WARING and EULER but was given in a more definit form first by D. BERNOULLI[1]). This idea was modified and developed into a method independently by G. DANDELIN[2]), N. I. LOBATSCHEWSKIJ[3]) and C. H. GRAEFFE[4]) and is called as method of GRAEFFE-BERNOULLI in the literature. This method runs as follows. Let

(1. 1) $$f_0(z)=a_{0,0}+a_{1,0}z+\ldots+a_{n,0}z^n, \qquad (a_{n,0}=1)$$

and we suppose its zeros $z_1, z_2, \ldots, z_n$ satisfie the inequality

(1. 2) $$|z_1|<|z_2|<\ldots<|z_n|.$$

We define the polynomials

(1. 3) $$f_\nu(z)=a_{0,\nu}+a_{1,\nu}z+\ldots+a_{n,\nu}z^n, \qquad (a_{n,\nu}=1)$$

by

(1. 4) $$f_{\nu+1}(z)=(-1)^n f_\nu(\sqrt{z})\,f_\nu(-\sqrt{z})$$

---

[1]) D. BERNOULLI, Commentationes Petropolitanae. Vol. 3.

[2]) G. DANDELIN, Recherches sur la resolution des équations numériques. *Nouveaux mém. de l'acad. roy. des sciences et belles lettres de Bruxelles.* **3** (1826), 1—71.

[3]) N. I. LOBACSEVSZKIJ, Algebra ili vücsiszlenie konecsnüh. (Kazan, 1834.)

[4]) C. H. GRAEFFE, Die Auflösung der höheren numerischen Gleichungen als Beantwortung einer von der kgl. Akad. der Wiss. zu Berlin aufgestellten Preisfrage (Zürich, 1837.) 1—44.

which is equivalent to the coefficient-recursion

$$a_{n-m,\nu+1} = 2a_{n,\nu}a_{n-2m,\nu} - 2a_{n-1,\nu}a_{n-2m+1,\nu} + \ldots +$$
$$+ (-1)^{m-1}a_{n-m+1,\nu}a_{n-m-1,\nu} + (-1)^m a^2_{n-m,\nu}, \qquad (m = 1, 2, \ldots, n).$$

Hence the coefficients $a_{k,\nu}$ can be easily computed. It follows from (1.4) that the zeros of $f_\nu(z)$ are exactly tha numbers $z_1^{2^\nu}, z_2^{2^\nu}, \ldots, z_n^{2^\nu}$. Hence

$$\sum_{j=1}^{n} z_j^{2^\nu} = -a_{n-1,\nu}, \sum_{1\leqq j_1 < j_2 \leqq n} (z_{j_1}z_{j_2})^{2^\nu} = a_{n-2,\nu},$$

etc.; i. e. from (1.2) we have

$$|z_n| = \lim_{\nu\to\infty} |a_{n-1,\nu}|^{2^{-\nu}}, \quad |z_{n-1}z_n| = \lim_{\nu\to\infty} |a_{n-2,\nu}|^{2^{-\nu}},$$

or

$$|z_{n-1}| = \lim_{\nu\to\infty} \left|\frac{a_{n-2,\nu}}{a_{n-1,\nu}}\right|^{2^{-\nu}}.$$

Generally for $k = 1, 2, \ldots, n$ we have

$$(1.5) \qquad |z_k| = \lim_{\nu\to\infty} \left|\frac{a_{k-1,\nu}}{a_{k,\nu}}\right|^{2^{-\nu}} \equiv r_k$$

i. e. the absolute values of the zeros are determined. If we want the zeros themselves we have only to expand $f_0(z)$ into a Taylor-series around $z = h$ where h is so small that

$$|z_1 - h| < |z_2 - h| < \ldots < |z_n - h|$$

and apply the rule (1.5). Denoting the corresponding r_k-values by $r_k(h)$ the points of intersection of the circles

$$|z| = r_k; \ |z - h| = r_k(h)$$

leave for z_k only two possibities, the wrong one of whose can be easily removed.

2. Let us consider the first-part of the above sketched method which refers to the approximative values of $|z_k|$ only. What are the disadvantages, theoretical or practical of the method? The first theoretical disadvantage is that if $|z_1| = |z_2|$, it is no more true. If e. g. (α to be determined)

$$(2.1) \qquad f_0(z) = (z-1)(z-e^{i\alpha})(z-e^{-i\alpha}),$$

then we have obviously

$$-a_{2,\nu} = 1 + e^{2^\nu i\alpha} + e^{-2^\nu i\alpha} = 1 + 2\cos 2^\nu \alpha.$$

Choosing $\nu = 2l+1$, $\alpha = \frac{\pi}{3}$ we have

$$2^{2l+1} \equiv 2 \pmod 6$$
$$\cos\frac{2^\nu \pi}{3} = \cos\frac{(6v+2)\pi}{3} = \cos\frac{2\pi}{3} = -\frac{1}{2} \quad \text{and}$$

$$a_{2,\,2l+1} = 0 \qquad\qquad (l = 0, 1, 2, \ldots)$$

i. e. if the limes for $|z_n|$ would exist then it would be 0 which does not give the right value of $|z_n|$ which is[5]) 1. A practical disadvantage is presented by the fact that there is no method to decide whether not (1. 2) is satisfied. It means a further practical disadvantage that even if (1. 2) is fulfilled we have no rule to decide whether or not for a certain $\nu = \nu_0$ the quantity

$$\left| \frac{a_{k-1,\,\nu_0}}{a_{k,\,\nu_0}} \right|^{2^{-\nu_0}}$$

is "near enough" to the right value r_k. To a given arbitrarily large positive ω and arbitrarily small positive ε one can easily modify the example (2. 1) replacing $e^{\pm i\alpha}$ by $(1-\delta)\,e^{i\alpha}$ and $(1-2\delta)\,e^{-i\alpha}$ with a suitable positive $\delta\left(<\frac{1}{10}\right)$ so that there is a $\nu > \omega$ such that

$$|a_{2,\,\nu}| \leq \varepsilon$$

which is "far" from the right value 1.

3. Hence the practical value of the method seemed to be very small inspite of all efforts[6]) and even the theoretical basis of it was cleared up only in 1913 by PÓLYA[7]). The books "Lehrbuch der Algebra" of FRICKE (1924) and "Vorlesungen über Algebra" of BAUER—BIEBERBACH (1929) showed no progress whatsoever. After the results of REY PASTOR incorporated in his "Lecciones de Algebra" (II. edition, 1932) the results of R. SAN JUAN[8]) meant the first essential progress. His idea was found independently and clearly by A. OSTROWSKI[9]) who resumed the question in 1940 in a paper of fundamental importance. Here really all phases of the approximationprocess are thoroughly analysed. He succeeded in getting rid of all of the above-mentioned defects by a modification of the method. Instead of the coefficients $a_{j,\nu}$ (ν fixed, $j = 0, 1, \ldots, n$) he introduced in this theory as a new element the notion of the Newton-majorant of the polynomial $f_\nu(z)$ (which occurs previously in a disguised form in the quoted papers[8]) of R. SAN JUAN), i. e. the polynom

---

[5]) A still simpler and clearer counter-example is given by Prof. G. HAJÓS. Taking $f_0(z) = z^3 - 1$ is is evidently $f_\nu(z) \equiv f_0(z)$, $\nu = 1, 2, \ldots$, i. e. $a_{2,\,\nu} = 0$ for *all* ν*'s* while $|z_3| = 1$.

[6]) See e. g. in the first volume of Encyklopedie der math. Wiss. the survey of C. RUNGE, Separation und Approximation der Wurzeln.

[7]) G. PÓLYA, Über das Graeffesche Verfahren. *Zeitschrift f. Math: u. Phys.* **63** (1915), 275—290.

[8]) R. SAN-JUAN, Complementos al método de Gräffe para la resolucion de ecuaciones algébricas. *Revista Matematica Hispano-Americana.* Ser. **3** I. (1939) and Compléments a la méthode de Gräffe pour la résolution des équations algébriques. *Bull. des Sciences Math.* **LIX.** (1935), 104—109.

[9]) A. OSTROWSKI, Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent. *Acta Math.* **72** (1940), 99—257.

$$M(z, f_\nu) = \sum_{j=0}^{n} T_{j,\nu} z^j$$

which is uniquely determined by the following three postulats:

1) $$|a_{j,\nu}| \leq T_{j,\nu} \qquad (j = 0, 1, \ldots, n).$$

2) Putting $R_{j,\nu} = \frac{T_{j-1,\nu}}{T_{j,\nu}}$ we require

$$R_{j,\nu} \leq R_{j+1,\nu} \qquad (j = 1, 2, \ldots, n)$$

3) If a polynomial

$$M^*(z, f_\nu) = \sum_{j=0}^{n} T^*_{j,\nu} z^j$$

satisfies 1) and 2) then

$$T_{j,\nu} \leq T^*_{j,\nu} \qquad (j = 0, 1, \ldots, n).$$

The existence of $M(z, f_\nu)$ follows easily though the explicit determination of the $T_{j,\nu}$ quantities is somewhat cumbersome either by calculation or graphically. We have

$$T_{0,\nu} = |a_{0,\nu}|$$

and, owing to $a_{n,\nu} = 1$,

$$T_{n,\nu} = 1.$$

Then he proved if the zeros of $f_0(z)$ are

(3. 1) $$|z_1| \leq |z_2| \leq \ldots \leq |z_n|$$

(what can be supposed without loss of generality) then as approximating value of $|z_k|$ the value

$$\left(\frac{T_{k-1,\nu}}{T_{k,\nu}}\right)^{2^{-\nu}} = R_{k,\nu}^{2^{-\nu}}$$

can be chosen. More exactly he proved the inequalities

(3. 2) $$\left(1 - 2^{-\frac{1}{k}}\right)^{2^{-\nu}} \leq \frac{|z_k|}{R_{k,\nu}^{2^{-\nu}}} \leq \left(1 - 2^{-\frac{1}{n-k+1}}\right)^{-2^{-\nu}} \qquad (k = 1, 2, \ldots n).$$

The quantities on the both tails of (3. 2) tend to 1 if $\nu \to \infty$. It is remarkable that these two quantities i. e. the rapidity of convergence do not depend upon the coefficients of $f_0(z)$, it depends only upon n and k which was notized by SAN JUAN.[8])

We write out explicitely (3. 2) for $k = n$; this asserts that

$$\left(1 - 2^{-\frac{1}{n}}\right)^{2^{-\nu}} \leq \frac{|z_n|}{(T_{n-1,\nu})^{2^{-\nu}}} \leq 2^{2^{-\nu}},$$

what he[10]) improved to

$$n^{-2^{-\nu}} \leq \frac{|z_n|}{(T_{n-1,\nu})^{2^{-\nu}}} \leq 2^{2^{-\nu}}. \tag{3.3}$$

4. In the later years I developed a method which I used in various questions of the theory of RIEMANN's zeta-function, of the theory of gap-series, of quasi-analytic functions and in other topics.[11]) The basis of the method is the theorem that having the complex numbers $w_1, w_2, \ldots, w_N$ with

$$|w_1| \leq |w_2| \leq \ldots \leq |w_N| = w_N = 1 \tag{4.1}$$

then denoting the power-sum

$$w_1^j + w_2^j + \ldots + w_N^j$$

by σ_j we have for $m \geq N$ the inequality

$$\max_{m \leq j \leq m+N} |\sigma_j| \geq \left(\frac{N}{e^6 m}\right)^N \tag{4.2}$$

independently of the configuration of the w_j's.

Before I succeeded in proving this theorem I raised the question as a prelude, what lower estimation can be given supposing (4. 1) to

$$\max_{1 \leq j \leq N} |\sigma_j|.$$

I first proved the inequality

$$\max_{1 \leq j \leq N} |\sigma_j| \geq \frac{1}{N}. \tag{4.3}$$

A modification of the idea of this proof lead P. ERDŐS to the estimation

$$\max_{1 \leq j \leq N} |\sigma_j| \geq \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{N}\right)^{-1}, \tag{4.4}$$

I proved finally the inequality

$$\max_{1 \leq j \leq N} |\sigma_j| \geq \log 2 \left(\frac{1}{1} + \ldots + \frac{1}{N}\right)^{-1}. \tag{4.5}$$

The inequality (4. 5) is slightly better then (4. 4). Having no applications in view at that time I did not much care with my further conjecture according which there is a numerical c_1 constant such that from (4. 1)

$$\max_{1 \leq j \leq N} |\sigma_j| \geq c_1 \tag{4.6}$$

follows. I communicated this conjecture among others to my very talented late pupil, *N.* SCHWEITZER who was killed by the Nazis on the 28. Jan. 1945

[10]) L. c. [9]) Theorem IX. p. 143.

[11]) Compare my lecture at the meeting of the Tschechoslovakian and Polish Mathematical Associatiens in Prague on 2. Sept. 1949 entitled "On a new method in the analysis with applications" *Casopis pro pěst. mat. a fys.* **74** (1949) 123–131. — These and new applications will be given in a forthcoming book.

being only 22 years old; he proved my conjecture in the important case when all the quantities σ_j are real, with[12]) $c_1 = 1$. Also the corollary of this theorem is due to him, according which we have for the general case

$$\max_{1 \leq j \leq 2N} |\sigma_j| \geqq \frac{1}{2}. \tag{4.7}$$

This will be incorporated in Lemmas II and III. Now I observed recently that all the results (4. 2), (4. 3), (4. 4), (4. 5), (4. 7) imply modifications of the method of DANDELIN—LOBATSCHEVSKIJ—GRAEFFE, more or less fit to actual numerical computation. A proof of my conjecture (4. 6) (which is true perhaps with $c_1 = \frac{1}{2}$ generally) seems to be very desirable now since it would give considerable simplifications in carrying out the calculations. I intend to return to this subject elsewhere, but I remark right now that my conjecture is in the general case not true with $c_1 = 1$; even in the case $N = 2$ taking e. g.

$$w_1 = \frac{1}{2} e^{\frac{2\pi i}{3}}$$

we obtain after a little geometrical consideration

$$|1 + w_1| < 1, \quad |1 + w_1^2| < 1.$$

This shows that the systems $(w'_1, \ldots, w'_N)$ for whose

$$\max (|\sigma_1|, |\sigma_2|, \ldots, \sigma_N|) = \text{minimal}$$

are such that some of the quantities w'_ν are inside the unit-circle, against the expectation; by the way it is easy to show

$$\max (|\sigma_1|, |\sigma_2|, \ldots, |\sigma_N|) \geqq 1.$$

if all the quantities w_ν are absolutely $\geqq 1$.

5. My method gives approximative values principially for all $|z_k|$'s but with the notation (3. 1) the approximations for $1 < k < n$ are uncomparably worse than those of OSTROWSKI's method; so we restrict ourselves to the cases $k = 1$ and $k = n$, i. e. to the case of the zeros with minimal and maximal absolute value respectively. Obviously it is enough to consider the case $k = n$. A comparative analysis of both methods for this case will follow later; that the approximation of the zeros with largest absolute value deserves so much attention is shown by two facts. Firstly for the approximation of all other zeros can be reduced to the successive applications of it. Secondedly it is of practical use. As Prof. E. EGERVÁRY kindly informed me, e. g. having n wheels rotating on an axe it is very important to determine their least critical angular-velocity and this is given by the smallest zero of an algebraic equation of the degree $2n$.

---

[12]) As the example $w_N = 1, w_1 = w_2 = \ldots = w_{N-1} = 0$ shows this estimation cannot be improved.

6. We shall give three procedures for obtaining approximating values for the case $k=n$. The first of these is given by the

First rule. *If the zeros of the equation*

$$f_0(z)=a_{0,0}+a_{1,0}z+\ldots+a_{n,0}z^n=0 \qquad (a_{n,0}=1) \tag{6.1}$$

to be solved are $z_1, z_2, \ldots, z_n$ *with*

$$|z_1|\leq|z_2|\leq\ldots\leq|z_n| \tag{6.2}$$

then first we form the transforms $f_\mu(z)$ *by the recursion*

$$f_{\mu+1}(z)=(-1)^n f_\mu(\sqrt{z})f_\mu(-\sqrt{z}),\quad \mu=0,1,\ldots,\nu-1. \tag{6.3}$$

Denoting

$$f_\nu(z)=\sum_{j=0}^{n} a_{j,\nu}z^j, \qquad (a_{n,\nu}=1) \tag{6.4}$$

and the j^{th} *power-sum of its zeros by* $s_{j,\nu}$ *we compute further the quantities* $s_{1,\nu}, s_{2,\nu}, \ldots, s_{2n,\nu}$ *by the* NEWTON–GIRARD *formulas:*

$$\begin{aligned}
s_{1,\nu}+a_{n-1,\nu}&=0\\
s_{2,\nu}+a_{n-1,\nu}s_{1,\nu}+2a_{n-2,\nu}&=0\\
&\vdots\\
s_{n,\nu}+a_{n-1,\nu}s_{n-1,\nu}+\ldots+na_{0,\nu}&=0\\
s_{n+1,\nu}+a_{n-1,\nu}s_{n,\nu}+\ldots+a_{0,\nu}s_{1,\nu}&=0\\
&\vdots\\
s_{2n,\nu}+a_{n-1,\nu}s_{2n-1,\nu}+\ldots+a_{0,\nu}s_{n,\nu}&=0.
\end{aligned} \tag{6.5}$$

Then we have[13])

$$n^{-2^{-\nu}}\leq\frac{|z_n|}{\left(\max\limits_{j=1,2,\ldots,2n}|s_{j,\nu}|^{\frac{1}{j}}\right)^{2^{-\nu}}}\leq 2^{2^{-\nu}}. \tag{6.6}$$

7. Compared to OSTROWSKI's bounds (3.3) our bounds (6.6) are, curiously enough, exactly the same; an improvement of lemma III. would give narrower bounds. As to the calculation both methods form first the transforms (6.3); the difference comes afterwards. Our first rule prescribes then the recursive forming of the sequence $s_{j,\nu}$ by (6.5) — a process which needs only the first three fundamental operations and this can be quickly performed by any type of computing machines — and then $2n$ extractions of root which is a longer operation. OSTROWSKI's procedure requires however the formation of the NEWTON-majorant which needs at least n and at most n^2 extraction of root. If my conjecture expressed in **4** is true then in (6.6) the right side could be replaced by $c_1^{-2^{-\nu}}$ and — what is the really important — the appro-

[13]) The simplicity and limitation of the number of operations necessary to obtain e.g. a precision of 1% suggest that the method can perhaps serve as a basis for a computing machine.

ximative value for $|z_n|$ in (6. 6) could be replaced by

$$(7.1) \qquad (\max_{j=1,\ldots,n} |s_{j,\nu}|^{\frac{1}{j}})^{2^{-\nu}},$$

which would mean a considerable reduction of the calculations, first of all would lessen the number of extractions of root to the half of the original number.

There is a type of problems at which the rule (6. 6) may be more advantageous than OSTROWSKI's method. It is well-known from the theory of spectra that a multiplett line of a spectrum will be decomposed in magnetic fields. The mathematical treatment of this phenomenon leads to a secular equation of higher degree the matrix of which is symmetrical (i. e. all zeros are real) but the coefficients depend upon parameters. If e. g. the coefficients are rational functions of the parameter t then replacing t by different values the Newton diagramm may change time to time. Following the rule (6. 6) the quantities $|s_j|^{\frac{1}{j}}$ are functions of t easy to draw and their upper enveloppe will give the dependence of $|z_n|$ upon t with a controllable error.

8. What can be said about the approximation by the value (7. 1) without any conjecture? At the present I can prove only the

Second rule. *For the equation* (6. 1) — *using the notation* (6. 2), (6. 3), (6 4) — *we form the first* ν *transforms* (6. 3) *and afterwards the first* n *power-sums* $s_{j,\nu}$ *only, by the recursion* (6. 5). *Then we have*

$$(8.1) \qquad n^{-2^{-\nu}} \leq \frac{|z_n|}{(\max_{j=1,\ldots,n} |s_{j,\nu}|^{\frac{1}{j}})^{2^{-\nu}}} \leq \left(\frac{1}{\log 2}\left(\frac{1}{1}+\ldots+\frac{1}{n}\right)\right)^{2^{-\nu}}.$$

9. Further we mention a third rule which is the least practical but was chronologically the first.

Third rule. *With an integer* $m \leq n$ *and the notations of the first rule we from the quantites* $s_{m,\nu}, s_{m+1,\nu}, \ldots, s_{m+n,\nu}$. *Then we have*

$$n^{-\frac{1}{m}} \leq \frac{|z_n|}{(\max_{m \leq j \leq m+n} |s_{j,\nu}|^{\frac{1}{j}})^{\frac{1}{m}}} \leq \left(\frac{e^6 m}{n}\right)^{\frac{n}{m}}.$$

If $m \to \infty$ then both tails $\to 1$ though slowly.

10. Now I turn to a related subject which claimes however to have only theoretical interest at the present. As far as I know there is no known method for the determination of the absolutely greatest imaginary part of the zeros of a polynomial. A way is opened to such one by the following

Theorem I. *We consider the equation*

$$(10.1) \qquad a_0 + a_1 z + \cdots + a_n z^n = 0$$

with the zeros $z_1, z_2, \ldots, z_n$ *for which we suppose*

$$|Iz_1| \leq |Iz_2| \leq \cdots \leq |Iz_{n-1}| < |Iz_n|. \tag{10.2}$$

If $H_m(z)$ *stands for the* m^{th} *Hermite-polynomial which is defined by*

$$e^{-z^2} H_m(z) = (-1)^n (e^{-z^2})^{(m)}, \tag{10.3}$$

then for the expression

$$U_m = \sum_{\nu=1}^{n} H_m(z_\nu), \tag{10.4}$$

which is evidently expressible explicitely by $a_0, a_1, \ldots, a_n$, *the limes-relation*

$$\lim_{m\to\infty} \frac{1}{\sqrt{2m}} \log \left\{ \frac{\Gamma\left(1+\frac{m}{2}\right)}{\Gamma(1+m)} |U_m| \right\} = |Iz_n| \tag{10.5}$$

holds.

I shall return to this subject later.

11. In his paper[9]) (§. 7) OSTROWSKI gives bounds for $|z_1 z_2 \ldots z_l|$ where — with a little changed notation — we denote by $z_1, z_2, \ldots, z_m$ the zeros of

$$f(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + a_m z^m = 0 \qquad (a_m = 1) \tag{11.1}$$

with

$$|z_1| \leq |z_2| \leq \cdots \leq |z_l| \leq \cdots \leq |z_m|. \tag{11.2}$$

He compares $|z_1 z_2 \ldots z_l|$ with the coefficients of the Newton-majorant of $f(z)$; if this is denoted again by

$$M(z, f) = \sum_{j=0}^{m} T_j z^j,$$

then with a little change of his notation he proved

$$\frac{1}{2l+1} \leq \frac{|z_1 z_2 \ldots z_l|}{\frac{T_0}{T_l}} \leq \binom{m}{l}. \tag{11.3}$$

Using our lemmas I found that we can form with the coefficients a_j another expression which gives narrower bounds for $|z_1 z_2 \ldots z_l|$. To achieve this we form the polynomials

$$f_{(k)}(z) = \sum_{j=0}^{m} a_{j,k} z^j \qquad (a_{m,k} = 1) \tag{11.4}$$

$$k = 1, 2, \ldots, 2\binom{m}{l}$$

where $f_{(k)}(z)$ has the zeros

$$z_1^k, z_2^k, \ldots, z_m^k.$$

Of course the coefficients $a_{j,k}$ can be expressed by the a_j's. Then the role played by $\frac{T_0}{T_l}$ will be given to

$$\max_{1 \leq k \leq 2\binom{m}{l}} |a_{l,k}|^{\frac{1}{k}}.$$

More exactly we shall prove the

Theorem II. *With the above notation we have instead of* (11. 3)

$$(11.5) \qquad \frac{1}{2} \leq \frac{|z_1 z_1 \ldots z_l|}{\max\limits_{1 \leq k \leq 2\binom{m}{l}} |a_{l,k}|^{\frac{1}{k}}} \leq \binom{m}{l}.$$

Applying Theorem II. instead of $f(z)$ to its ν^{th} transform (6. 3), the bounds can be made as near to 1 as we wish.

12. Now we turn to the proofs of some lemmas.

Lemma I. *Let*

$$w_1^j + \cdots + w_N^j = \sigma_j$$

and — what is no loss of generality —

$$w_N = |w_N| = \max_{1 \leq j \leq N} |w_j| = 1,$$

then we have

$$M = \max_{1 \leqq j \leqq N} |\sigma_j| \geqq \log 2 \left(\frac{1}{1} + \ldots + \frac{1}{N}\right)^{-1}.$$

To prove this we consider the equation

$$(12.1) \qquad w^N + b_1 w^{N-1} + \ldots + b_N = 0$$

whose zeros are exactly our w_j-numbers. NEWTON—GIRARD's formulae give

$$(12.2) \qquad \sigma_l + b_1 \sigma_{l-1} + \ldots + b_{l-1}\sigma_1 + l b_l = 0 \qquad (l = 1, 2, \ldots, N).$$

Now I start from the first identity of (12. 2). Then we have

$$|b_1| = |\sigma_1| \leq M = \binom{M}{1}.$$

Then from the second identity of (12. 2)

$$2|b_2| = |\sigma_2 + b_1\sigma_1| \leqq |\sigma_2| + |b_1||\sigma_1| \leqq M + M^2,$$

$$|b_2| \leqq \binom{M+1}{2}.$$

We suppose as proved already for all $r < h \leqq N$

$$(12.3) \qquad |b_r| \leqq \binom{M+r-1}{r}.$$

Then applying the h^{st} identity of (12. 2) we obtain

$$h|b_h| \leqq |\sigma_h| + |b_1||\sigma_{h-1}| + \ldots + |b_{h-1}||\sigma_1| \leqq$$

$$\leqq M \left\{1 + \binom{M}{1} + \binom{M+1}{2} + \ldots + \binom{M+h-2}{h-1}\right\} = M\binom{M+h-1}{h-1},$$

i. e.

$$|b_h| \leqq \binom{M+h-1}{h}$$

hence (12. 3) is true for $r=h$, i. e. for $1 \leqq h \leqq N$. Since $w=1$ is a zero of (12. 1) we have

$$1 = -(b_1+b_2+\ldots+b_N) = |b_1+\ldots+b_N| \leqq |b_1|+|b_2|+\ldots+|b_N| \leqq$$

(12. 4) $$\leqq \binom{M}{1}+\binom{M+1}{2}+\ldots+\binom{M+N-1}{N} = \binom{M+N}{N}-1$$

Since

$$\binom{M+N}{N} = \left(1+\frac{M}{1}\right)\left(1+\frac{M}{2}\right)\ldots\left(1+\frac{M}{N}\right) < e^{M\sum_{j=1}^{N}\frac{1}{j}}$$

i. e. from (12. 4)

$$2 < e^{M\sum_{j=1}^{N}\frac{1}{j}}.$$

Q. e. d.

13. Now we turn to the

Lemma II. *Let $w_1, w_2, \ldots, w_N$ be such that with a w_j also $\bar{w}_j$ occours among the w-numbers and one of them, say w_N, is positive and $\geqq 1$. Then*

(13. 1) $$M \equiv \max_{1 \leqq j \leqq N} |\sigma_j| \geqq 1.$$

SCHWEITZER's proof for this lemma starts with the same idea i. e. forming the equation

(13. 2) $$w^N + b_1 w^{N-1} + \ldots + b_N = 0$$

with the zeros $w_1, w_2, \ldots, w_N$ and considering the power-sums σ_j together with the b_j's by means of NEWTON—GIRARD formulae. The additional condition of the numbers w_j means simply that all the coefficients b_j are real. He uses induction with respect to N. For $N=1$ the assertion is evident. We suppose it is proved for all $m<N$ that is among the numbers

$$w_1, w_2, \ldots, w_m$$

a complex number can occur only together with its conjugate and e. g. w_m is positive and $\geqq 1$, then

$$\max_{1 \leqq j \leqq m} |w_1^j + \ldots + w_m^j| \geqq 1$$

and consider the case $m=N$. First we make the apparent restriction

(13. 3) $$w_N = 1.$$

Then we form the equation (13. 2) and NEWTON—GIRARD formulas

$$\sigma_j + b_1 \sigma_{j-1} + \ldots + b_{j-1}\sigma_1 = -j b_j \qquad (j=1, 2, \ldots, N)$$

all the b_j being real numbers. Summing all these we obtain

(13. 4) $$\sigma_N + \sum_{l=1}^{N-1} \sigma_{N-l}(1+b_1+\ldots+b_l) = -\sum_{l=1}^{N} l b_l = -\sum_{l=1}^{N}(b_{N-l+1}+\ldots+b_N).$$

Using (13. 3) we have

$$-(b_{N-l+1}+\ldots+b_N)=1+b_1+\ldots+b_{N-l};$$

hence (13. 4) gives

$$\sigma_N+\sum_{l=1}^{N-1}\sigma_{N-l}(1+b_1+\ldots+b_l)=1+\sum_{l=1}^{N-1}(1+b_1+\ldots+b_l)$$

or with the notation of (13. 1)

$$M\geq\frac{\left|1+\sum_{l=1}^{N-1}(1+b_1+\ldots+b_l)\right|}{1+\sum_{l=1}^{N-1}|1+b_1+\ldots+b_l|}\,. \tag{13. 5}$$

Now we distinguish two cases.

I. All the quantities

$$1+b_1+\ldots+b_l$$
$$l=1,2,\ldots,(N-1)$$

are non-negative. Then the assertion follows at once from (13. 5).

II. There is an integer s such that $1\leq s\leq N-1$ and

$$1+b_1+\ldots+b_s<0. \tag{13. 6}$$

Than we consider the equation

$$G(w)=w^s+b_1w^{s-1}+\ldots+b_s=0$$

with the zeros $w_1', w_2', \ldots, w_s'$ and real coefficients. (13. 6) gives together with $\mathrm{sg}\,G(+\infty)>0$ that $G(w)$ has a real zero >1. This fact, together with the other one that the reality of the b-coefficients gives that with a w_j' also $\overline{w_j'}$ occurs among the w_j'-roots, implies that our induction hypothesis can be applied to the quantities w_j' and thus

$$\max_{1\leq j\leq s}|\sigma_j'|\equiv\max_{1\leq j\leq s}|w_1'^j+\ldots+w_s'^j|\geq 1. \tag{13. 7}$$

But writing out the NEWTON-GIRARD formulas for $G(w)$ and for the equation (13. 2) we see

$$\sigma_j'=\sigma_j \qquad (j=1,2,\ldots,s).$$

Hence

$$M=\max_{1\leq j\leq N}|\sigma_j|\geq\max_{1\leq j\leq s}|\sigma_j|=\max_{1\leq j\leq s}|\sigma_j'|\geq 1 \tag{13. 8}$$

i. e. lemma II is proved with the restriction (13. 3).

If we require instead of (13. 3) only that there is a positive $w_N\geq 1$ then we have only to introduce the quantities α_ν by

$$w_\nu=w_N\,\alpha_\nu \qquad \nu=1,2,\ldots,N. \tag{13. 9}$$

Then $\alpha_N=1$ and owing to the positivity of w_N the numbers α_ν share with the w_j's the property that together with a complex number also the conjugate occurs among them. Hence (13. 8) is applicable to the α_j's; hence

$$\max_{1\le j\le N}|\sigma_j| = \max_{1\le j\le N}|w_N|^j|\alpha_1^j+\ldots+\alpha_N^j| \ge \max_{1\le j\le N}|\alpha_1^j+\ldots+\alpha_N^j| \ge 1. \quad \text{Q. e. d.}$$

14. Lemma III is a simple corollary of Lemma II.

Lemma III. *If* $\xi_1, \xi_2, \ldots \xi_n$ *are such that*

$$|\xi_n| = \max_{1\le l\le n}|\xi_l| \ge 1$$

then we have

$$\max_{1\le j\le 2n}|\xi_1^j+\ldots+\xi_n^j| \ge \frac{1}{2}.$$

For the proof we introduce the quantities β_ν by

$$\xi_\nu = \xi_n\beta_\nu \qquad \nu = 1, 2, \ldots, n.$$

Then we have

$$\beta_n = 1.$$

Then applying Lemma II with $N = 2n$ and

$$w_1 = \beta_1, w_2 = \beta_2, \ldots, w_n = \beta_n$$
$$w_{n+1} = \bar\beta_1, w_{n+2} = \bar\beta_2, \ldots, w_{2n} = \bar\beta_n$$

we have

$$2\max_{1\le j\le 2n}|\beta_1^j+\beta_2^j+\ldots+\beta_n^j| \ge 2\max_{1\le j\le 2n}|R(\beta_1^j+\ldots+\beta_n^j)| =$$
$$= \max_{1\le j\le 2n}|\beta_1^j+(\bar\beta_1)^j+\ldots+\beta_n^j+(\bar\beta_n)^j| \ge 1.$$

Hence

$$\max_{1\le j\le 2n}|\xi_1^j+\xi_2^j+\ldots+\xi_n^j| = \max_{\le j\le 2n}|\xi_n|^j|\beta_1^j+\ldots+\beta_n^j| \ge$$
$$\ge \max_{1\le j\le 2n}|\beta^j+\ldots+\beta_n^j| \ge \frac{1}{2}. \qquad \text{Q. e. d.}$$

15. Finally we mention the

Lemma IV. *If* $m \ge N$ *and*

$$\max_{1\le j\le N}|w_j| \ge 1$$

then we have[14])

$$\max_{m\le j\le m+N}|w_1^j+\ldots+w_N^j| \ge \left(\frac{N}{e^6 m}\right)^N.$$

---

[14]) In a slightly weaker form see my paper "On Riemann's hypothesis" *Bull. de l'Acad. des Sciences de l'URSS.* **11** (1947) 197.–262. lemma XII. For the proof of the present form see my forthcoming paper "On Carlson's theorem" (to be published in *Acta Mat. Acad. Scientiarum Hungaricae*).

16. Now we turn to the proof of our rules. We shall prove only the first rule; the proofs of the two remaining rules differ from that of the first only by employing Lemma I resp. Lemma IV instead of Lemma III.[15])

To prove the first rule we consider an arbitrary power-sum $s_{j,\nu}$ $1 \leq j \leq 2n$. For this we have using (6. 2)

$$|s_{j,\nu}| = \Big|\sum_{k=1}^{n} z_k^{j\cdot 2^\nu}\Big| \leq n|z_n|^{j\cdot 2^\nu}$$

i. e.

$$\frac{|z_n|}{(|s_{j,\nu}|^{\frac{1}{j}})^{2^{-\nu}}} \geq n^{-\frac{1}{j2^\nu}} \geq n^{-2^\nu}. \tag{16. 1}$$

Taking for j that j_0, for which

$$\max_{1\leq j\leq 2n} |s_{j,\nu}|^{\frac{1}{j}}$$

is attained, (16. 1) gives

$$\frac{|z_n|}{(\max\limits_{1\leq j\leq 2n} |s_{j,\nu}|^{\frac{1}{j}})^{2^{-\nu}}} \geq n^{-2^\nu}. \tag{16. 2}$$

To prove the upper estimation we apply Lemma III with

$$\xi_l = \left(\frac{z_l}{z_n}\right)^{2^\nu} \qquad (l = 1, 2, \ldots, n).$$

The condition of this lemma is obviously fulfilled; hence there is an integer k with $1 \leq k \leq 2n$ and

$$\left|\left(\frac{z_1}{z_n}\right)^{2^\nu.k} + \left(\frac{z_2}{z_n}\right)^{2^\nu.k} + \ldots + \left(\frac{z_n}{z_n}\right)^{2^\nu.k}\right| \geq \frac{1}{2}$$

or

$$|z_n| \leq 2^{\frac{1}{k.2^\nu}}(|s_{k,\nu}|^{\frac{1}{k}})^{2^{-\nu}} \leq 2^{2^{-\nu}}(\max_{1\leq k\leq 2n} |s_{k,\nu}|^{\frac{1}{k}})^{2^{-\nu}}$$

$$\frac{|z_n|}{(\max\limits_{1\leq k\leq 2n} |s_{k,\nu}|^{\frac{1}{k}})^{2^{-\nu}}} \leq 2^{2^{-\nu}}. \tag{17. 3}$$

(17. 2) and (17. 3) prove the first rule.

17. Now we prove the theorem formulated in **10.** The proof is based upon the formula[16])

---

[15]) The analogous application of Lemma II gives another rule relating to equations with real coefficients only, which fulfill moreover the unusual condition that they have among their zeros with maximal absolute value also a real one. Then the upper bound in (6. 6) can be replaced by 1 instead of $2^{2^{-\nu}}$.

[16]) See e. g. G. Szegő: Orthogonal polynomials. *Amer. Math. Soc. Coll. Publ.* XXIII (1939) esp. p. 197.

$$\lim_{m\to\infty}\frac{1}{\sqrt{2m}}\log\left\{\frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma(m+1)}|H_m(z)|\right\}=|Iz| \tag{17.1}$$

valid for all fixed non-real z-values and the integral-representation[17])

$$e^{-z^2}H_m(z)=(-1)^{\left[\frac{m}{2}\right]}\frac{2^{m+1}}{\sqrt{\pi}}\int_0^\infty e^{-t^2}t^m \begin{matrix}\cos\\ \sin\end{matrix}\, 2zt\,dt \tag{17.2}$$

if m is even resp. odd. It follows from (17.2) for real z's

$$|e^{-z^2}H_m(z)|\leq\frac{2^{m+1}}{\sqrt{\pi}}\int_0^\infty e^{-t^2}t^m dt=\frac{2^m}{\sqrt{\pi}}\Gamma\left(\frac{m+1}{2}\right). \tag{17.3}$$

The condition (10.2) means that z_m is not real. Then we write from (10.4)

$$|U_m|=H_m|(z_n)|\left|1+\sum_{\nu=1}^{n-1}\frac{H_m(z_\nu)}{H_m(z_n)}\right|$$

and thus we obtain

$$\frac{1}{\sqrt{2m}}\log\left\{\frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma(m+1)}|U_m|\right\}=\frac{1}{\sqrt{2m}}\log\left\{\frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma(m+1)}|H_m(z_n)|\right\}+$$

$$+\frac{1}{\sqrt{2m}}\log\left|1+\sum_{\nu=1}^{n-1}\frac{H_m(z_\nu)}{H_m(z_n)}\right|. \tag{17.4}$$

The first term on the right $\to|Iz_n|$ using (17.1). Hence to complete the proof of the theorem we have only to show that the second term in (17.4) tends to 0 if $m\to\infty$. We consider the expression

$$J=\sum_{\nu=1}^{n-1}\frac{H_m(z_\nu)}{H_m(z_n)}=\sum_{z_\nu \text{ complex}}+\sum_{z_\nu \text{ real}} \tag{17.5}$$

(17.1) can be written in the form

$$|H_m(z)|=\frac{\Gamma(m+1)}{\Gamma\left(\frac{m}{2}+1\right)}e^{\sqrt{2m}\,|Iz|\,(1+o(1))}, \tag{17.6}$$

where the o-sign refers to $m\to\infty$ with a fixed z if z is complex. If $1\leq\nu<n$ and z_ν is complex, (17.6) gives

$$\left|\frac{H_m(z_\nu)}{H_m(z_n)}\right|=e^{\sqrt{2m}(1+o(1))\,(|Iz_\nu|-|Iz_n|)}.$$

Hence

$$\left|\sum_{z_\nu \text{ complex}}\right|\leq n e^{-\sqrt{2m}(1+o(1))\min\limits_{1\leq\nu\leq n-1}(|Iz_n|-|Iz_\nu|)}\to 0. \tag{17.7}$$

---

[17]) Ibid. p. 103.

Further from (17.3) for real z-values

$$|H_m(z)| \leq \frac{2^m}{\sqrt{\pi}} e^{z^2} \Gamma\left(\frac{m+1}{2}\right).$$

Hence if z_j is real then

$$\left|\frac{H_m(z_j)}{H_m(z_n)}\right| \leq \frac{2^m}{\sqrt{\pi}} \frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m+1)} e^{z_j^2 - \sqrt{2m}\,|Iz_n|\,(1+o(1))}. \tag{17.8}$$

If m is even then we have

$$\frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m+2}{2}\right)}{\Gamma(m+1)} < \frac{\Gamma\left(\frac{m+2}{2}\right)^2}{\Gamma(m+1)} = \frac{1}{\binom{m}{\frac{m}{2}}} < \frac{1}{\frac{2^m}{m+1}} = \frac{m+1}{2^m}.$$

If n is odd then

$$\frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m+1)} = (m+1)\,\frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m+}{2}\right)}{\Gamma(m+2)} <$$

$$< (m+1)\frac{\Gamma\left(\frac{m+3}{2}\right)^2}{\Gamma(m+2)} < \frac{(m+1)\,(m+2)}{2^{m+1}}.$$

Hence for all integer m values we have

$$\frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m+1)} < \frac{(m+2)^2}{2^m}$$

and from (17.8)

$$\left|\frac{H_m(z_j)}{H_m(z_n)}\right| < \frac{1}{\sqrt{\pi}}(m+2)^2 e^{z_j^2 - \sqrt{2m}\,|Iz_n|\,(1+o(1))}.$$

From this we have evidently for $m \to \infty$

$$\left|\sum_{z_j \text{ real}} \frac{H_m(z_j)}{H_m(z_n)}\right| \to 0.$$

This and (17.7) give that $J \to 0$ for $m \to \infty$ and hence the second term in (17.4) tends to 0 indeed. This completes the proof of the Theorem I.

18. Now we turn to the proof of the Theorem II. With the notation (11.1), (11.2), (11.4) we have to prove (11.5) for $1 \leq l \leq m$. For the proof we shall proceed similarly as at the proof of our rules. Going over to the reciprocal polynomial we have to show again with the notation (11.1), (11.2), (11.4)

$$\frac{1}{\binom{m}{l}} \leq \frac{|z_{m-l+1} z_{m-l+2} \cdots z_m|}{\max\limits_{1 \leq k \leq 2\binom{m}{l}} |a_{m-l,k}|^{\frac{1}{k}}} \leq 2. \tag{18.1}$$

First of all we have for all integer $k \geq 1$

(18.2) $$|a_{m-l,k}| = \left| \sum_{1 \leq i_1 < i_2 < \ldots < i_l \leq m} (z_{i_1} \ldots z_{i_l})^k \right| \leq \binom{m}{l} |z_{m-l+1} \ldots z_m|^k.$$

If we take $k = k_0$, where k_0 is that index k for which

$$\max_{1 \leq k \leq 2\binom{m}{l}} |a_{m-l,k}|^{\frac{1}{k}}$$

is attained then (18.2) gives

$$\frac{|z_{m-l+1} \ldots z_m|}{|a_{m-l,k_0}|^{\frac{1}{k_0}}} \geq \binom{m}{l}^{-\frac{1}{k_0}} \geq \binom{m}{l}^{-1}$$

which is the first half of our assertion (18.1). To prove the second part we apply again Lemma III with

$$n = \binom{m}{l}, \ \zeta_j = \frac{z_{i_1} z_{i_2} \ldots z_{i_l}}{z_{m-l+1} \ldots z_m}, \ 1 \leq i_1 < i_2 < \cdots < i_l \leq m$$

The condition of this lemma is obviously satified. Hence we have an integer k_1, with $1 \leq k_1 \leq 2\binom{m}{l}$ such that

$$\left| \frac{\sum_{1 \leq i_1 < i_2 < \ldots < i_l \leq m} (z_{i_1} \ldots z_{i_l})^{k_1}}{(z_{m-l+1} \ldots z_m)^{k_1}} \right| \geq \frac{1}{2}$$

or

$$\frac{|a_{m-l,k_1}|}{(z_{m-l+1} \ldots z_m)^{k_1}} \geq \frac{1}{2}.$$

Hence

$$|z_{m-l+1} \ldots z_m| \leq 2^{\frac{1}{k_1}} |a_{m-l,k_1}|^{\frac{1}{k_1}} \leq 2 \max_{\substack{1 \leq k \leq 2\binom{m}{l} \\ k \text{ integer}}} |a_{m-l,k}|^{\frac{1}{k}}.$$

This completes the proof of the Theorem II.

(Received November 8, 1950.)