On some problems concerning Poisson processes.

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Introduction.

Let us consider a stochastic process of random events, which is of MARKOV's type (strictly speaking: a differential process), but not necessarily homogeneous in time. Let the process start at time $t=0$, and let the random variable ζ_t $(t>0)$ denote the number of events which occur in the time interval $(0,t)$. The following suppositions are made:

A) if $t_1<t_2\leqq t_3<t_4\leqq\cdots\leqq t_{2k-1}<t_{2k}$ $(k=2,3,\ldots)$, the random variables $\zeta_{t_2}-\zeta_{t_1}, \zeta_{t_4}-\zeta_{t_3},\ldots,\zeta_{t_{2k}}-\zeta_{t_{2k-1}}$ are independent of each other.

B) Let $w_k(s,t)$ denote the probability of exactly k events occurring in the time interval (s,t) $(s<t;\ k=0,1,2,\ldots)$: we suppose that for any arbitrary small $\varepsilon>0$ and any arbitrary large $T>0$ a positive number $\delta>0$ can be found such that if $t_1<t_2\leqq t_3<t_4\leqq\cdots<t_{2n}<T$ and $\sum_{r=1}^{n}(t_{2r}-t_{2r-1})<\delta$ we have $\prod_{r=1}^{n} w_0(t_{2r-1},t_{2r})>1-\varepsilon$; by other words, if the total length of the intervals (t_{2r-1},t_{2r}), $(r=1,2,\ldots,n)$ does not exceed δ, the probability that no event will take place in any of these intervals is greater than $1-\varepsilon$ (here n is an arbitrary positive integer). Condition *B)* postulates the "rarity" of the events considered in that sense that it is highly probable that no event will take place during a sufficiently short time consisting of any number of time intervals[1]). We shall refer to *B)* as the first postulate of rarity.

C) We suppose further that for every $s\geqq 0$

$$\lim_{\substack{\Delta s\to 0\\ \Delta s>0}}\frac{w_1(s,s+\Delta s)}{1-w_0(s,s+\Delta s)}=1.$$

Condition *C)* postulates the "rarity" of the events considered, in a different sense: it states that if the length the time intervall considered tends to 0,

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[1]) This can be expressed also by saying that the "interval function" $1-w_0(s,t)$ is absolutely continuous.

the probability of the occurrence of at least one event is in the limit equal to the probability of the occurrence of exactly one event or, by other words, that the probability of the occurrence of more than one event in a short time interval is in the limit negligible compared with the probability of the occurrence of one event. We shall refer to *C)* as the second postulate of rarity. In § 1. we shall prove that under conditions *A)*, *B)* and *C)* the process is a POISSON process, i. e. there exists a nonnegative, *L*-mesurable function $\lambda(\tau)$ such that putting

$$\Lambda(t)=\int_0^t \lambda(\tau)\,d\tau \tag{1}$$

we have

$$w_k(s,t)=\frac{(\Lambda(t)-\Lambda(s))^k}{k!}e^{-(\Lambda(t)-\Lambda(s))} \qquad (s\leq t;\; k=0,1,2,\ldots). \tag{2}$$

The proof is similar to the proof given in the homogeneous case in a joint paper of L. JÁNOSSY, J. ACZÉL and the author of the present paper[2]). In the homogeneous case, i. e. if $w_k(s,t)$ depends only on $t-s$, the condition *B)* is clearly not necessary, and it follows from conditions *A)* and *C)* that $\Lambda(t)=\lambda t$ and therefore

$$w_k(s,t)=\frac{[\lambda(t-s)]^k}{k!}e^{-\lambda(t-s)} \tag{3}$$

where $\lambda>0$ is a constant.

In § 2 we discuss the following problem: let us suppose that every event in a POISSON process is the starting point of a happening, which has a definite duration, this duration being also a random variable, the distribution law of which may depend on the time when the happening started. Let us denote by $F(t,\tau)$ probability that a happening which started at time t is finished before $t+\tau$, i. e. has a duration $<\tau$. Let us denote by η_t the number of happenings going on at time t; clearly η_t is a random variable and we may ask about its distribution law. Let $p_K(t)$ denote the probability of exactly K happenings going on at time t. Let us put further

$$\varphi(t,\tau)=1-F(t,\tau). \tag{4}$$

We shall prove that if the RIEMANN—STIELTJES integral

$$A(t)=\int_0^t \varphi(\tau,t-\tau)\,d\Lambda(\tau) \tag{5}$$

exists for every t, — thus especially if $\lambda(\tau)$ and $\varphi(\tau,t-\tau)$ are continuous functions — we have

$$p_k(t)=\frac{[A(t)]^k}{k!}e^{-A(t)} \tag{6}$$

[2]) L. JÁNOSSY, A. RÉNYI, J. ACZÉL: On composed Poisson distributions. *Acta Math. Hung.*, **1** (1951), in print.

where $A(t)$ is defined by (5). By other words η_t has also a POISSON distribution. Thus for instance if the underlying POISSON process is homogeneous in time, i. e. $\Lambda(t) = \lambda t$, $(\lambda > 0)$ and if $F(t, \tau) = 1 - e^{-\mu\tau}$, where $\mu > 0$, we have

$$A(t) = \frac{\lambda}{\mu}(1 - e^{-\mu t}) \tag{7}$$

and thus

$$p_k(t) = \frac{\left[\frac{\lambda}{\mu}(1 - e^{-\mu t})\right]^k}{k!} e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})} \tag{8}$$

a result which is well known[3]).

Note that it follows from (8) that if $t \to \infty$, the distribution of η_t tends to a limiting distribution:

$$\lim_{t \to \infty} p_k(t) = v_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} e^{-\frac{\lambda}{\mu}}. \tag{9}$$

The above problem has many interesting physical and technical applications. We mention only three of them.

a) Let us consider some radioactive substance A; if an atom of A disintegrates, we shall say that an event occurred, and denote by ζ_t the number of atoms which disintegrated in the time interval $(0, t)$. If an A-atom disintegrates, an other atom, — say a B-atom — originates. Let us suppose that the B-atoms are also radioactive; in this case a "happening" is the existence of a B-atom, which starts at the moment when a B-atom originates, and is finished at the moment when the B-atom considered disintegrates; in this case η_t — the number of "happenings going on" — denotes simply the number of B-atoms present at time t. The existence of $\lim\limits_{t \to \infty} p_k(t)$ expresses the well known fact that the quantity of B-atoms present tends to a limit, which is expressed in physics by saying that after some time an equilibrium is reached. (9) shows that the small fluctuations about this equilibrium follow POISSON's law. Of course (8) and (9) can be applied to this problem only if it is physically reasonable to suppose $\lambda(\tau)$ constant, (i. e. if it can be neglected that the quantity of A-atoms decreases in time). If this is not the case, we have to put $\lambda(t) = \lambda e^{-\nu t}$, (i. e. take into account the exponential decrease in the quantity of A-atoms owing to disintegrations) and we obtain

$$p_k(t) = \frac{\left(\lambda \frac{e^{-\nu t} - e^{-\mu t}}{\mu - \nu}\right)^k}{k!} e^{-\lambda \frac{e^{-\nu t} - e^{-\mu t}}{\mu - \nu}} \qquad (k = 0, 1, 2, \ldots). \tag{10}$$

In this case we have naturally $\lim\limits_{t \to \infty} p_k(t) = 0$ if $(k = 1, 2, \ldots$ and $\lim\limits_{t \to \infty} p_0(t) = 1$;

[3]) A. JENSEN: An elucidation of Erlang's statistical works through the theory of stochastic processes. (The Life and Works of A. K. Erlang) Copenhagen, 1948, pp. 23—100.

this means that in the limit there remains no B-atom at all (and of course no A-atom either).

b) Let us consider a telephone centre, with a practically infinite number of telephone lines. If a telephone call is made, we shall say that an event occurred, and denote by ζ_t the number of calls during the time interval $(0, t)$; every call is the starting point of a conversations, the duration of which depends on chance. In this case a conversation going on through the centre is called a happening, and η_t denotes the number of conversations going on at time t. In telephone engineering it is generally supposed that during the most busy hours of a day, the density of calls is constant, i. e. the process is homogeneous in time; it is maintained further by many specialists that the length of conversations have an exponential law of distribution $F(t, \tau) = 1 - e^{-\mu\tau}$. It seems however that this assumption is used in first place because of its simplicity: as a matter of fact the random variables η_t form a differential process if (and only if) $F(t, \tau) = 1 - e^{-\mu\tau}$; indeed in this case the probability that a conversation going on at time t will be finished in the time interval $(t, t + \Delta t)$ does not depend on that how long the conversation considered has been going on before, because of

$$\frac{d_\tau F(t, \tau)}{1 - F(t, \tau)} = \mu \tag{11}$$

being constant[4]). Let us mention, that our results enable to treat the question mentioned above without supposing the density of calls being constant, and by taking into account that the distribution of the length of conversations may depend also on the hour of day (which seems rather reasonable).

c) Let us consider a vacuum tube, and let us say that an event occurred, when an electron leaves the cathod; thus ζ_t denotes the number of electrons which left the cathod during the time interval $(0, t)$; the happening started by an event is in this case the flight of the electron in the tube, and η_t denotes the number of electrons in the tube at time t. The underlying Poisson process is in this case generally not homogeneous in time; as a matter of fact $\lambda(t)$ depends on the temperature of the cathod. The function $F(t, \tau)$ is in this case of a rather intricate nature[5]), it depends on many factors, especially on the average speed of the electrons which leave the cathod and on the voltage mpressed on the grid from outside and depends thus in general explicity on t also.

In all these applications (and in other ones not mentioned here) our results make it possible to investigate the situation under fairly general conditions.

I have announced the results of § 2 recently in a lecture, at the Hungarian Academy of Sciences together with a heuristic approach instead of proof.

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[4]) Conversely if we suppose that (11) holds, it follows that $F(t, \tau) = 1 - e^{-\mu\tau}$.

[5]) We suppose conditions in which the effect of the space charge can be neglected.

In the present paper these results are proved rigorously by the use of the method of generating functions.

§ 1. The non-homogeneous Poisson process.

As stated in the introduction, we shall prove that if conditions *A)*, *B)* and *C)* are satisfied, we have (2).

It follows from *A)* that

$$w_0(0,t) = w_0(0,s)\, w_0(s,t) \qquad (s \leq t), \tag{1.1}$$

thus putting $\Lambda(t) = -\log w_0(0,t)$, we have

$$w_0(s,t) = e^{-(\Lambda(t)-\Lambda(s))}. \tag{1.2}$$

As the left hand side of (1.2) is ≤ 1, it follows that $\Lambda(t)$ is non-decreasing. Condition *B)* implies that for any $\varepsilon > 0$ and $T > 0$ there can be found a $\delta > 0$ such that if $t_1 < t_2 < \cdots < t_{2n} < T$ and $\sum_{r=1}^{n} (t_{2r} - t_{2r-1}) < \delta$

$$\sum_{r=1}^{n} (\Lambda(t_{2r}) - \Lambda(t_{2r-1})) < \log \frac{1}{1-\varepsilon}. \tag{1.3}$$

This means that $\Lambda(t)$ is absolutely continuous in the interval $(0, T)$ and thus can be represented in the form

$$\Lambda(t) = \int_0^t \lambda(\tau)\, d\tau \quad \text{with} \quad \lambda(\tau) \geq 0. \tag{1.4}$$

Thus (2) is proved for $k = 0$. Next it follows from *A)* that

$$w_1(0,t) = w_0(0,s)\, w_1(s,t) + w_1(0,s)\, w_0(s,t) \tag{1.5}$$

or, putting

$$v_1(s,t) = w_1(s,t)\, e^{(\Lambda(t)-\Lambda(s))}$$

we have

$$v_1(s,t) = v_1(0,t) - v_1(0,s). \tag{1.6}$$

Let us denote $v_1(0,t) = \Lambda_1(t)$, it follows from (1.6) that $v_1(s,t) = \Lambda_1(t) - \Lambda_1(s)$ and thus, taking (1.6) into account, we obtain

$$w_1(s,t) = (\Lambda_1(t) - \Lambda_1(s))\, e^{-(\Lambda(t)-\Lambda(t))}. \tag{1.7}$$

As $w_1(s,t) \leq 1 - w_0(s,t)$, it follows from *B)* using the inequality $1 - e^{-x} < x$ for $x > 0$ and putting $e^{\Lambda(T)} = K$

$$\sum_{r=1}^{n} [\Lambda_1(t_{2r}) - \Lambda_1(t_{2r-1})] \leq K \sum_{r=1}^{n} w_1(t_{2r-1}, t_{2r}) \leq K \sum_{r=1}^{n} [\Lambda(t_{2r}) - \Lambda(t_{2r-1})] \tag{1.8}$$

and thus $\Lambda_1(t)$ is also absolutely continuous, and we may put

$$\Lambda_1(t) = \int_0^t \lambda_1(\tau)\, d\tau.$$

Now let us apply condition *C)*; it follows that

$$\lambda_1(\tau) = \lambda(\tau) \qquad \text{almost ewerywhere,} \tag{1.9}$$

and thus $\Lambda_1(t) \equiv \Lambda(t)$, and therefore

$$w_1(s, t) = [\Lambda(t) - \Lambda(s)]\, e^{-[\Lambda(t) - \Lambda(s)]} \tag{1.10}$$

which proves (2) for $k = 1$.

Let us suppose, that (2) is proved for $k \leq n - 1$, we shall prove that it holds also for $k = n$; thus (2) follows by induction for all values of k. We start from the identity

$$w_n(0, t) = \sum_{k=0}^{n} w_k(0, s)\, w_{n-k}(s, t) \qquad (s < t). \tag{1.11}$$

Substituting the values of $w_k(0, s)$ and $w_k(s, t)$ for $k = 0, 1, 2, \ldots, (n-1)$ from (2) into (1.11) it follows

$$\begin{aligned} w_n(0, t) &= w_n(0, s)\, e^{-(\Lambda(t) - \Lambda(s))} + w_n(s, t)\, e^{-\Lambda(s)} + \\ &+ \frac{\Lambda^n(t) - [\Lambda(t) - \Lambda(s)]^n - \Lambda^n(s)}{n!}\, e^{-\Lambda(t)}. \end{aligned} \tag{1.12}$$

Putting

$$f(s, t) = w_n(s, t)\, e^{\Lambda(t) - \Lambda(s)} - \frac{[\Lambda(t) - \Lambda(s)]^n}{n!} \tag{1.13}$$

we obtain from (1.12) $f(s, t) = f(0, t) - f(0, s)$ and thus

$$w_n(s, t) = \left[\frac{[\Lambda(t) - \Lambda(s)]^n}{n!} + f(0, t) - f(0, s)\right] e^{-[\Lambda(t) - \Lambda(s)]}. \tag{1.14}$$

Now as $w_n(s, t) \leq 1 - w_0(s, t)$, the interval function $w_n(s, t)$ is absolutely contiouous, and thus $f(0, t) - f(0, s) = \int_s^t g(\tau)\, d\tau$; It follows from condition *C)* that

$$\lim_{\Delta s \to 0} \frac{w_n(s, s + \Delta s)}{w_1(s, s + \Delta s)} = 0 \qquad \text{for } n = 2, 3, \ldots$$

and thus we obtain $g(\tau) = 0$ almost ewerywhere, and therefore from (1.14) it results

$$w_n(s, t) = \frac{(\Lambda(t) - \Lambda(s))^n}{n!}\, e^{-(\Lambda(t) - \Lambda(s))} \tag{1.16}$$

and (2) is proved for $k = n$ also.

§ 2. The distribution of η_t.

First of all let us introduce certain notations. Let us divide the interval $(0, t)$ into n equal parts by means of the points $t_k = \frac{kt}{n}$ $(k = 0, 1, 2, \ldots, n)$ and let us put $t_k - t_{k-1} = \Delta t_k$, $\Lambda(t_k) - \Lambda(t_{k-1}) = \Delta \Lambda_k$ $(k = 1, 2, \ldots, n)$. If $F(t, \tau)$ denotes the distribution functions of the length of a happening which started at time t, and $\Phi(t, \tau) = 1 - F(t, \tau)$ let us put

(2. 1) $$M_k = \operatorname*{Max}_{t_{k-1} \leq \tau \leq t_k} \Phi(\tau, t-\tau), \quad m_k = \operatorname*{Min}_{t_{k-1} \leq \tau \leq t_k} \Phi(\tau, t-\tau).$$

Let $U_k(r)$ denote the probability of that there are exactly r such happenings going on at time which started in the time interval (t_{k-1}, t_k), $(r = 0, 1, 2, \ldots)$. We s'art from the inequalities

(2. 2) $$\sum_{s=r}^{\infty} \binom{s}{r} w_s(t_{k-1}, t_k)\, m_k^r\, (1 - M_k)^{s-r} \leq U_k(r)$$
$$U_k(r) \leq \sum_{s=r}^{\infty} \binom{s}{r} w_s(t_{k-1}, t_k)\, M_k^r\, (1 - m_k)^{s-r}$$

which are fundamental for the proof which we shall develop here. (2. 2) is obtained as follows: if r happenings are going on at time t which all started in the time interval (t_{k-1}, t_k), there must have been $s \geq r$ events in this interval; if a happening started exactly at time τ, $t_{k-1} \leq \tau \leq t_k$, the probability that it will continue going on at time t, is $\varphi(\tau, t-\tau)$; as we do not know the value of τ exactly we can state only that this probability lies somewhere between m_k and M_k; similarly the probability that the happening considered is finished before t is equal to $1 - \varphi(\tau, t-\tau)$ and thus its value lies somewhere between $1 - M_k$ and $1 - m_k$; thus (2. 2) follows. Now let us introduce the generating function of the probabilities $U_k(r)$:

(2. 3) $$\psi_k(z) = \sum_{r=0}^{\infty} U_k(r) z^r.$$

Here and in what follows we consider the generating functions only for real values of z lying between 0 and 1. If $0 < z < 1$ it follows from (2. 2) that

(2. 4) $$e^{\Delta \Lambda_k (z m_k - M_k)} \leq \psi_k(z) \leq e^{\Delta \Lambda_k (z M_k - m_k)}$$

which may be also written in the form

(2. 5) $$\psi_k(z) = e^{\Delta \Lambda_k \cdot M_k (z-1) + \vartheta_k \Delta \Lambda_k (M_k - m_k)}$$

where $-1 \leq \vartheta_k \leq +1$.

Now if $p_N(t)$ denotes the probability of exactly N happenings going on at time t, we have evidently

(2. 6) $$p_N(t) = \sum_{r_1 + r_2 + \ldots + r_n = N} U_1(r_1)\, U_2(r_2) \cdots U_n(r_n)$$

where the summation is extended over all (ordered) n-tuples of nonnegative integers $r_1, r_2, \ldots, r_n$ such that $\sum_{K=1}^{n} r_k = N$. It follows, introducing the generating function

(2. 7) $$\pi(z) = \sum_{N=1}^{\infty} p_N(t) z^N$$

that

(2. 8) $$\pi(z) = \prod_{k=1}^{n} \psi_k(z),$$

and thus from (2.5)

$$\pi(z) = e^{\left(\sum_{k=1}^{n} M_k \cdot \Delta \Lambda_k\right)(z-1) + \vartheta\left(\sum_{k=1}^{n} (M_k - m_k)\Delta \Lambda_k\right)} \tag{2.9}$$

where $-1 \leq \vartheta \leq +1$. Let now n tend to ∞; if the RIEMANN—STIELTJES integral $\int_0^t \varphi(\tau, t-\tau)\, d\Lambda(\tau)$ exists, or by other words, if $\varphi(\tau, t-\tau)$ is Riemann integrable with respect to the weigth function $\Lambda(\tau)$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} M_k \cdot \Delta\Lambda_k = \int_0^t \varphi(\tau, t-\tau)\, d\Lambda(\tau), \tag{2.10}$$

and (because the oscillation sums are tending to 0)

$$\lim_{n\to\infty} \sum_{k=1}^{n} (M_k - m_k)\, \Delta\Lambda_k = 0. \tag{2.11}$$

Thus it follows from (2.9) that

$$\pi(z) = e^{\left(\int_0^t \varphi(\tau, t-\tau)\, \lambda(\tau)\, d\tau\right)(z-1)}. \tag{2.12}$$

Developing the expression at the right of (2.12) according to the powers of z, and comparing the coefficients at both sides of (2.12) we obtain

$$p_n(t) = \frac{\left(\int_0^t \varphi(\tau, t-\tau)\, \lambda(\tau)\, d\tau\right)^n}{n!}\, e^{-\int_0^t \varphi(\tau, t-\tau)\, \lambda(\tau)\, d\tau}. \tag{2.13}$$

Thus it is proved that η_t follyws also a POISSON distribution.

Finally it should be mentioned, that the distribution of η_t can be determined by the method developed above also in case the underlying differential process (the process ζ_t) is not a POISSON process, but is of a more general type. We shall return to this question at an other occasion.

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(Received March 29, 1950.)