On groups every subgroup of which is a direct summand.

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In what follows we determine all groups every subgroup of which is a direct summand. It will be found that such a group G is necessarily abelian, and the order of each element of G is a square-free number. This implies that G contains no element of order 0 (i. e., with another terminology, "of infinite order"). Conversely: an arbitrary abelian group all elements of which are of square-free order has the property mentioned above, by the following

Theorem:[1]) *Each subgroup of a group G is a direct summand of G if and only if G is abelian, and the order of each element of G is a square-free number.*

We write the groups additively even if they are not necessarily abelian. Groups will be denoted by capital letters and their elements by Greek letters. Ordinary integers will be denoted by small Latin letters. In particular, p denotes always a prime number.

Now we are going to prove the theorem.

a) Let G be an arbitrary group having each of its subgroups as a direct summand. First we prove G to be abelian. Let α and β be two arbitrary elements of G; by our hypothesis we have $G=\{\alpha\}+H$, where $\{\alpha\}$ denotes the cyclic group generated by α. Hence $\beta=n\alpha+\eta$, $\eta\in H$. On the other hand $\alpha+\eta=\eta+\alpha$ by the property of the direct sum, and therefore we obtain

$$\alpha+\beta=\alpha+n\alpha+\eta=n\alpha+\alpha+\eta=n\alpha+\eta+\alpha=\beta+\alpha.$$

We show, moreover, that the order of each element α of G is a square-free number (consequently $\neq 0$). Indeed, if p is an arbitrary prime number, $G=\{p\alpha\}+K$ holds for a suitable $K\subset G$. Then we have

[1]) The author believes this theorem to be new, and in the contrary case he hopes to be excused by the shortness and simplicity of his proof which makes no use of any results of the theory of groups.

$\alpha = mp\alpha + \varkappa$ $(\varkappa \in K)$ i. e, by $\{p\alpha\} \cap K = 0$,

$$(mp-1)p\alpha = p\varkappa = 0.$$

Hence it follows that the order of α is a positive number not divisible by p^2.

b) Now we prove the converse statement. Let A be an abelian group each element of which is of square-free order. If B is an arbitrary subgroup of A, we have to show $A = B + C$ with a suitable $C \subset A$. For this purpose we define C as a *maximal* subgroup of A having the property

$$B \cap C = 0.$$

(The existence of such a group C is assured by ZORN's lemma.) Then $\{B, C\} = B + C$ holds; and therefore we have only to show $B + C = A$.

Assume $B + C \neq A$. Then there exists an element $\alpha \in A$ with

(1) $$\alpha \bar{\in} B + C, \quad p\alpha \in B + C$$

for a suitable prime number p. Therefore we have

(2) $$p\alpha = \beta + \gamma \qquad (\beta \in B, \gamma \in C).$$

The order of $p\alpha$, by our assumption, being not divisible by p, each member of (2) is annihilated by a number prime to p. Hence there exist numbers u, v such that $\beta = pu\beta$, $\gamma = pv\gamma$. So we have by (1) and (2)

$$\alpha - u\beta - v\gamma = \alpha' \bar{\in} B + C, \quad p\alpha' = 0.$$

Therefore $(B + C) \cap \{\alpha'\} = 0$, i. e. $B \cap (C + \{\alpha'\}) = 0$ which contradicts the choice of C. This completes the proof of the theorem.

Remark. Thanks are due to L. FUCHS for having remarked that in the theorem we may replace "all subgroups of G" by "all cyclic subgroups of G" without altering its validity. — It may also be noted that the corresponding problem on rings with unit element is solved in J. VON NEUMANN's work Continuous Geometry (Princeton, 1937), vol. 2, chapter 2. Indeed, Neumann proved that one of the characterizing properties of regular rings (with unit element) is that every principal right ideal has a corresponding „inverse" right ideal such that their direct sum (in the group-theoretic sense only) produces the whole ring.

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