Semilattices of nil-extensions
of rectangular groups

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Abstract. Some structural descriptions of semigroups which are semilattices of nil-extensions of rectangular groups were considered in [1], [3] and [4]. In this paper some new, indicatorial (i.e. in terms of forbidden members) and other descriptions for these semigroups will be given.

Decomposition of semigroups into a semilattice of completely Archimedean semigroups is a very important topic investigated in the papers of M. S. PUTCHA [16], M. L. VERONESI [22], and more intensive in the serials of papers of L. N. SHEVRIN [17], [18], [19], [20] and of the authors of this paper [1], [4], [5], [6]. If the components in these decompositions are orthodox (i.e. their idempotents form subsemigroups), then we have semilattices of nil-extensions of rectangular groups. These semigroups also form a very interesting class of semigroups. Recently X. TANG [21] solved the problem of characterization of semigroups with congruence extension property in this class. Some structural descriptions of semigroups which are semilattices of nil-extensions of rectangular groups were considered in [1], [3] and [4]. In this paper we will give some new, indicatorial (i.e. in terms of forbidden members) and other descriptions for these semigroups. Also, we will study chains of completely Archimedean semigroups and chains of nil-extensions of rectangular groups.

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers. If S is a semigroup, $\mathrm{Reg}(S)$ will denote the set of all regular elements of S, $E(S)$ the set of all idempotents of S, for $a \in \mathrm{Reg}(S)$, $V(a)$ the set of all *inverses* of a, i.e. $V(a) = \{x \in S \mid a = axa,\ x = xax\}$, and for $e \in E(S)$, G_e will denote the maximal subgroup of S with e as its identity.

A semigroup S is π-*regular* if for every $a \in S$ there exist $n \in \mathbf{Z}^+$, $x \in S$ such that $a^n = a^n x a^n$. A semigroup S is *completely π-regular* if for every $a \in S$ there exist $n \in \mathbf{Z}^+$, $x \in S$ such that $a^n = a^n x a^n$, $a^n x = x a^n$, i.e. if a^n lies in a subgroup of S. Note that L. N. SHEVRIN [20] used the name *epigroups* for these semigroups. If S is a completely π-regular semigroup, $x \in S$ and x^n belongs to some subgroup G of S, then x^0 will denote the identity of G.

A semigroup S with a zero 0 is a *nil-semigroup* if for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. An ideal extension S of a semigroup T is a *nil-extension* of T if the factor semigroup S/T is a nil-semigroup. A semigroup S is *Archimedean* if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in SbS$. A semigroup S is *completely Archimedean* if S is Archimedean and has a primitive idempotent, or equivalently, if it is a nil-extension of a completely simple semigroup.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be classes of semigroups. By $\mathcal{X}_1 \circ \mathcal{X}_2$ we denote the *Mal'cev product* of the classes $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e. the class of all semigroups S on which there exists a congruence ρ such that S/ρ is in $\mathcal{X}_2$ and each ρ-class which is a subsemigroup is in $\mathcal{X}_1$ [14]. If $\mathcal{X}$ is a class of semigroups, by $\mathbf{S}(\mathcal{X})$ and $\mathbf{H}(\mathcal{X})$ we will denote the class of all subsemigroups of semigroups from $\mathcal{X}$ and the class of all homomorphic images of semigroups from $\mathcal{X}$, respectively. If $\mathcal{X} = \{S\}$, then we write $\mathbf{S}(S)$ and $\mathbf{H}(S)$ instead of $\mathbf{S}(\mathcal{X})$ and $\mathbf{H}(\mathcal{X})$.

We introduce the following notations:

Notation	Class of semigroups	Notation	Class of semigroups
$\mathcal{CA}$	*completely Archimedean*	$\mathcal{M} \times \mathcal{G}$	*rectangular groups*
$\mathcal{S}$	*semilattices*	$\mathcal{C}$	*chains*
$\mathcal{C}\pi\mathcal{R}$	*completely π-regular*	$\mathcal{N}$	*nil-semigroups*

Clearly $\mathcal{X} \circ \mathcal{S}$ is the class of all semilattices of semigroups from the class $\mathcal{X}$. If $\mathcal{X}_2$ is a subclass of the class $\mathcal{N}$, then $\mathcal{X}_1 \circ \mathcal{X}_2$ is a class of all semigroups that are ideal extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$.

In this paper we will use the following semigroups given by the presentations:

$$A_2 = \left\langle a, e \mid a^2 = 0, e^2 = e, aea = a, eae = e \right\rangle,$$

$$B_2 = \left\langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b \right\rangle, \quad S_2 = \left\langle a, b \mid ab = ba \right\rangle,$$

and L_2 and R_2 will denote the twoelement left zero and right zero semigroups, respectively.

For undefined notions and notations we refer to [3], [8] and [11].

We start with the following construction: Let R be the ring $\mathbb{Z}$ of integers or the ring $\mathbb{Z}_p$ of residues mod p, $p \geq 2$, and let $I = \{0, 1\} \subseteq R$. Define a multiplication on $R \times I \times I$ by:

$$(m; i, \lambda)(n; j, \mu) = (m + n - (i - j)(\lambda - \mu); i, \mu),$$

$m, n \in R$, $i, j, \lambda, \mu \in I$. Then $R \times I \times I$ with this multiplication is a semigroup, and we will use the notations: $E(\infty) = \mathbb{Z} \times I \times I$, $E(p) = \mathbb{Z}_p \times I \times I$. The semigroup $E(\infty)$ $(E(p))$ is isomorphic to the Rees matrix semigroup over the additive group of the ring $\mathbb{Z}$ $(\mathbb{Z}_p)$ with the sandwich matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and it is not a rectangular group.

Theorem 1. *The following conditions on a semigroup S are equivalent:*

    (i)   *S is a semilattice of nil-extensions of rectangular groups;*

    (ii) *$S \in \mathcal{C}\pi\mathcal{R}$ and $(xy)^0 = (xy)^0(yx)^0(xy)^0$, for all $x, y \in S$;*

    (iii) *$S \in \mathcal{C}\mathcal{A} \circ \mathcal{S}$ and there are no $E(\infty)$ and $E(p)$, $p \in \mathbf{Z}^+$, $p \geq 2$, in $\mathbf{S}(S)$;*

    (iv) *$S \in \mathcal{C}\pi\mathcal{R}$ and there are no A_2, B_2, $E(\infty)$ and $E(p)$, $p \in \mathbf{Z}^+$, $p \geq 2$, in $\mathbf{S}(\mathbf{H}(S))$;*

    (v) *$S \in \mathcal{C}\pi\mathcal{R}$ and there are no A_2, B_2 and $E(p)$, $p \in \mathbf{Z}^+$, $p \geq 2$, in $\mathbf{H}(\mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R})$.*

PROOF. (i) $\Rightarrow$ (ii). Let S be a semilattice Y of semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$, let $S_\alpha \in (\mathcal{M} \times \mathcal{G}) \circ \mathcal{N}$. Clearly, $S \in \mathcal{C}\pi\mathcal{R}$. Assume $x, y \in S$. Then $xy, yx \in S_\alpha$, for some $\alpha \in Y$, whence $(xy)^0, (yx)^0 \in E(S_\alpha)$. Thus, $(xy)^0 = (xy)^0(yx)^0(xy)^0$, since $E(S_\alpha)$ is a rectangular band.

(ii) $\Rightarrow$ (iii). Let $a = axa$, $a, x \in S$. Then $ax, xa \in E(S)$, whence $(ax)^0 = ax$, $(xa)^0 = xa$, so by (ii) we obtain that $a = (ax)a = (ax)(xa)$ $(ax)a = ax^2a^2xa = ax^2a^2$. Thus, by Theorem 2.1 [1], $S \in \mathcal{C}\mathcal{A} \circ \mathcal{S}$, and by Theorem 2 [4], for every idempotent from S, each of its inverses is an idempotent. Now, if either one of the semigroups $E(\infty)$ or $E(p)$, $p \in \mathbf{Z}^+$, $p \geq 2$, is in $\mathbf{S}(S)$, then $(1; 0, 0)$ is an inverse of the idempotent $(0; 1, 1)$ in $E(\infty)$ $(E(p))$, and $(1; 0, 0)$ is not an idempotent. Thus, (iii) holds.

(iii) $\Rightarrow$ (i). To prove (i), it is sufficient to prove that every completely simple subsemigroup of S is a rectangular group. Let K be some completely simple subsemigroup of S. Assume that K is not a rectangular group, i.e. that there exists $e, f \in E(K)$ such that $ef \notin E(K)$. Thus, ef is a group element of K of order $p \geq 2$ or of infinite order, and it is easy to

verify that ef, efe, fef and fe are mutually different elements of the same (finite or infinite) order. Also, it can be checked directly that the mapping

$$(ef)^n \phi = (n; 0, 0), \quad (efe)^n \phi = (n; 0, 1),$$

$$(fe)^n \phi = (n; 1, 1), \quad (fef)^n \phi = (n; 1, 0),$$

is an isomorphism between the semigroup $\langle ef, fe \rangle$ and the semigroup $E(\alpha)$, where α is the order of ef (finite or infinite). Since this assertion is in contradiction with (iii), then we obtain that K is a rectangular group.

(iii) $\Rightarrow$ (iv). By Theorem [19], $S \in \mathcal{C}\pi\mathcal{R}$ and $A_2, B_2 \notin \mathbf{S}(\mathbf{H}(S))$, and by the proof of (iii) $\Rightarrow$ (i), $S \in ((\mathcal{M} \times \mathcal{G}) \circ N) \circ S$. Now, by Theorem 1 [4], Corollary 2 [2] and Theorem 2.1 [1], $\mathbf{H}(S) \subseteq ((\mathcal{M} \times \mathcal{G}) \circ N) \circ S$ and $\mathbf{S}(\mathbf{H}(S)) \subseteq \mathbf{S}(((\mathcal{M} \times \mathcal{G}) \circ N) \circ S)$, so there are no $E(\infty)$ and $E(p)$, $p \in \mathbf{Z}^+$, $p \geq 2$, in $\mathbf{S}(\mathbf{H}(S))$.

(iv) $\Rightarrow$ (iii). This follows by Theorem [19].

(i) $\Rightarrow$ (v). By Theorem [19], $S \in \mathcal{C}\pi\mathcal{R}$ and $A_2, B_2 \notin \mathbf{H}(\mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R})$. By Theorem 1 [9] ((i) $\Leftrightarrow$ (ii)), $\mathbf{S}(\mathcal{C}\mathcal{A}) \cap \mathcal{C}\pi\mathcal{R} \subseteq \mathcal{C}\mathcal{A}$, so it can be checked directly that $\mathbf{S}(\mathcal{X} = \mathcal{C}\mathcal{A} \circ \mathcal{S}) \cap \mathcal{C}\pi\mathcal{R} \subseteq \mathcal{X} = \mathcal{C}\mathcal{A} \circ \mathcal{S}$. By this and by Theorem 2.1 [1] ((i) $\Leftrightarrow$ (iii)), $\mathbf{S}(\mathcal{X}) \cap \mathcal{C}\pi\mathcal{R} \subseteq \mathcal{X}$, where $\mathcal{X} = ((\mathcal{M} \times \mathcal{G}) \circ \mathcal{N}) \circ \mathcal{S}$, whence

$$\mathbf{H}(\mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R}) \subseteq \mathbf{H}(\mathbf{S}(\mathcal{X}) \cap \mathcal{C}\pi\mathcal{R}) \subseteq \mathbf{H}(\mathcal{X}) \subseteq \mathcal{X},$$

so $E(p) \notin \mathbf{H}(\mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R})$, for any $p \in \mathbf{Z}^+$, $p \geq 2$. Thus, (v) holds.

(v) $\Rightarrow$ (iii). By Theorem [19], $S \in \mathcal{C}\mathcal{A} \circ \mathcal{S}$. Clearly, for each $p \in \mathbf{Z}^+$, $p \geq 2$, $E(p) \notin \mathbf{S}(S)$. Since $E(p) \in \mathbf{H}(E(\infty))$ for each $p \in \mathbf{Z}^+$, $p \geq 2$, then $E(\infty) \notin \mathbf{S}(S)$, by (v). Thus, (iii) holds. $\square$

Similarly we can prove the following

Theorem 2. *The following conditions on a semigroup S are equivalent:*

(i) S *is a semilattice of nil-extensions of left groups;*

(ii) $S \in \mathcal{C}\pi\mathcal{R}$ *and* $(xy)^0 = (xy)^0(yx)^0$, *for all* $x, y \in S$;

(iii) $S \in \mathcal{C}\pi\mathcal{R}$ *and there are no* A_2, B_2 *and* R_2 *in* $\mathbf{S}(\mathbf{H}(S))$;

(iv) $S \in \mathcal{C}\pi\mathcal{R}$ *and there are no* A_2, B_2 *and* R_2 *in* $\mathbf{H}(\mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R})$.

Corollary 1. *The following conditions on a semigroup S are equivalent:*

(i) S *is a semilattice of nil-extensions of groups;*

(ii) $S \in \mathcal{C}\pi\mathcal{R}$ *and* $(xy)^0 = (yx)^0$, *for all* $x, y \in S$;

(iii) $S \in \mathcal{C}\pi\mathcal{R}$ *and there are no* A_2, B_2, R_2 *and* L_2 *in* $\mathbf{S}(\mathbf{H}(S))$;

(iv) $S \in \mathcal{C}\pi\mathcal{R}$ and there are no A_2, B_2, R_2 and L_2
in $\mathbf{H}(\mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R})$.

Further we will consider chains of nil-extensions of rectangular groups. Some descriptions of these semigroups, and more generally, of chains of (completely) Archimedean semigroups were given by the authors in [1], [5] and [7]. Here we give some new characterizations of these semigroups. First we prove one more general theorem.

Theorem 3. *The following conditions on a semigroup S are equivalent:*

(i)   *S is a chain of completely Archimedean semigroups;*

(ii) *$S \in \mathcal{C}\pi\mathcal{R}$ and $\mathrm{Reg}(S)$ is a chain of completely simple semigroups;*

(iii) *$S \in \mathcal{C}\pi\mathcal{R}$ and there are no A_2, B_2 and S_2 in $\mathbf{H}(\mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R})$;*

(iv) *$S \in \mathcal{C}\pi\mathcal{R}$ and there are no A_2, B_2 and S_2 in $\mathbf{S}(\mathbf{H}(S))$.*

PROOF. (i) $\Rightarrow$ (ii). Let S be a chain Y of completely Archimedean semigroups S_α, $\alpha \in Y$, and for $\alpha \in Y$, let S_α be a nil-extension of a completely simple semigroup K_α.

Assume $a, b \in \mathrm{Reg}(S)$, $x \in V(a)$ and $y \in V(b)$. Then $xa, by \in E(S)$ and $xa \in S_\alpha$, $by \in S_\beta$, for some $\alpha, \beta \in Y$. Since Y is a chain, then $\alpha\beta = \alpha$ or $\alpha\beta = \beta$ Assume that $\alpha\beta = \alpha$. Then $xaby \in S_\alpha$ and $xa \in E(S_\alpha)$, so by Theorem 1 [9] and Lemma 1 [9], $xabyxa \in xaSxa = G_{xa}$, whence $xa \in xabyxaG_{xa}xabyxa \subseteq xabySbyxa$. Now

$$ab = axab \in axabySbyxab = abySbyxab \subseteq abSab.$$

Therefore, $ab \in \mathrm{Reg}(S)$. Similarly we prove that $\alpha\beta = \beta$ implies $ab \in \mathrm{Reg}(S)$. Hence, $\mathrm{Reg}(S)$ is a subsemigroup of S and clearly

$$\mathrm{Reg}(S) = \bigcup_{\alpha \in Y} \mathrm{Reg}(S_\alpha) = \bigcup_{\alpha \in Y} K_\alpha,$$

so $\mathrm{Reg}(S)$ is a chain Y of completely simple semigroups K_α, $\alpha \in Y$.

(ii) $\Rightarrow$ (i). Let $S \in \mathcal{C}\pi\mathcal{R}$ and let $\mathrm{Reg}(S)$ be a chain of completely simple semigroups. Then clearly $\mathrm{Reg}(S) = Gr(S)$, so by Theorem 1 [17], S is a semilattice of completely Archimedean semigroups and by Theorem 5.1 [1], S is a chain of completely Archimedean semigroups.

(iii) $\Rightarrow$ (i). By Theorem [19], S is a semilattice Y of completely Archimedean semigroups S_α, $\alpha \in Y$. If Y is not a chain, then Y contains a subsemigroup Z isomorphic to S_2, so $S_2 \in \mathbf{H}(T)$ and $T \in \mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R}$, where $T = \bigcup_{\alpha \in Z} S_\alpha$, which contradicts (iii). Thus, Y is a chain, so (i) holds.

(iv) $\Rightarrow$ (i). By Theorem [19], S is a semilattice Y of completely Archimedean semigroups S_α, $\alpha \in Y$. Since $Y \in \mathbf{H}(S)$, then $S_2 \notin \mathbf{S}(Y)$, by (vi), so Y is a chain.

(i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv). By Theorem [19], $S \in \mathcal{C}\pi\mathcal{R}$ and $A_2, B_2 \notin$ $\mathbf{H}(\mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R}) \cup \mathbf{S}(\mathbf{H}(S))$. Also, it can be checked directly that $\mathbf{H}(\mathcal{X}) \subseteq \mathcal{X}$ an $\mathbf{S}(\mathcal{X}) \cap \mathcal{C}\pi\mathcal{R} \subseteq \mathcal{X}$, where $\mathcal{X} = \mathcal{C}\mathcal{A} \circ \mathcal{C}$, whence $S_2 \notin \mathbf{H}(\mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R}) \cup$ $\mathbf{S}(\mathbf{H}(S))$. $\square$

Corollary 2. *The following conditions on a semigroup S are equivalent:*

(i)   *S is a chain of nil-extensions of rectangular groups;*

(ii)   *$S \in \mathcal{C}\pi\mathcal{R}$ and $\mathrm{Reg}(S)$ is a chain of rectangular groups;*

(iii) *$S \in \mathcal{C}\pi\mathcal{R}$ and $E(S)$ is a chain of rectangular bands;*

(iv) *$S \in \mathcal{C}\pi\mathcal{R}$ and there are no $A_2, B_2, E(\infty), E(p)$, $p \in \mathbf{Z}^+$, $p \geq 2$ and S_2 in $\mathbf{S}(\mathbf{H}(S))$;*

(v)   *$S \in \mathcal{C}\pi\mathcal{R}$ and there are no $A_2, B_2, E(p)$, $p \in \mathbf{Z}^+$, $p \geq 2$ and S_2 in $\mathbf{H}(\mathbf{S}(S) \cap \mathcal{C}\pi\mathcal{R})$.*

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