On direct sums of cyclic groups.

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An abelian group which is a direct sum of cyclic groups has evidently a minimal generating system, i. e. a generating system no element of which can be cancelled. The converse of this statement is not true, for there exist abelian groups — not direct sums of cyclic groups — having a minimal generating system. An example of such a group is the additive group of all rational numbers with square-free denominators. The set of the reciprocal values of all primes forms obviously a minimal generating system of this group.

In what follows we show that the existence of a special kind of minimal generating systems in abelian groups — which will be called an extremal generating system — involves the decomposibility of the group into a direct sum of cyclic groups. Our fundamental idea is taken from a recent paper[1]) of R. RADO containing a very nice proof of the basis theorem for finitely generated abelian groups. Thus the following theorem can also be considered as a generalization of RADO's result to the case of not necessarily finitely generated abelian groups.

The notions and symbols used are the following. The letters a, b denote elements of groups and the other small Latin letters ordinary integers. Groups are written additively. We denote by $O(a)$ the order of the element a of a group. Then $1 \leqq O(a) \leqq \infty$. An abelian group is *torsion free* if it contains no element $\neq 0$ of finite order. The symbol $\{a_1, a_2, \ldots\}$ denotes the group generated by the elements $a_1, a_2, \ldots$ of a group. Two systems $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ with the same finite number of elements of a group will be called *equivalent* if $\{a_1, \ldots, a_k\} = \{b_1, \ldots, b_k\}$. An arbitrary system S containing only elements $\neq 0$ of a group G we call an *extremal* system of G if S contains no finite subsystem $a_1, \ldots, a_k$ equivalent to a system $b_1, \ldots, b_k$ in G such that

$$\min_{1 \leqq i \leqq k} O(b_i) < \min_{1 \leqq i \leqq k} O(a_i).$$

[1]) R. RADO, A proof of the basis theorem for finitely generated Abelian groups. *Journ. London Math. Soc.* **26** (1951), **74**—75.

It is obvious that an extremal generating system (i. e. a generating system which is at the same time an extremal system) of a group is necessarily a minimal generating system.

Now we state the following

Theorem. *If an abelian group G has an extremal generating system S, then G is the direct sum of cyclic groups (generated by the elements of S).*

Corollary. *If a torsion free abelian group G has a generating system S such that each finite subsystem $a_1, \ldots, a_k$ of S is a generating system with the minimal number of elemenis of $\{a_1, \ldots, a_k\}$, then G is a direct sum of cyclic groups In particular, every generating system with the minimal number of elements of a finitely generated torsion free abelian group is a basis of the group.*

The corollary follows immediately from the theorem.

We base the proof of the theorem on the following lemma which is a slightly modificated form of RADO's lemma in[1]).

Lemma (of Rado). *If $r_1, \ldots, r_k$ are arbitrary integers with $(r_1, \ldots, r_k) = 1$, then any system $a_1, \ldots, a_k$ of elements of an abelian group G is equivalent to a system $b_1, \ldots, b_k$ in G, such that $b_1 = r_1 a_1 + \cdots + r_k a_k$.*

The conclusion of the lemma obviously holds if $s = |r_1| + \cdots + |r_k| = 1$. Thus in the case $s > 1$ we can use induction with respect to s. From $s > 1$ and $(r_1, \ldots, r_k) = 1$ it follows that at least two of the r_i are different from zero. Let $|r_1| \geq |r_2| > 0$. Then $|r_1 \pm r_2| < |r_1|$, i. e.

$$|r_1 \pm r_2| + |r_2| + \cdots + |r_k| < s$$

for one of the two signs. Thus by $(r_1 \pm r_2, r_2, \ldots, r_k) = 1$ and by the induction hypothesis we have

$$\{a_1, \ldots, a_k\} = \{a_1, a_2 \mp a_1, a_3, \ldots, a_k\} = \{b_1, \ldots, b_k\}$$

where

$$b_1 = (r_1 \pm r_2) a_1 + r_2(a_2 \mp a_1) + r_3 a_3 + \cdots + r_k a_k = r_1 a_1 + \cdots + r_k a_k,$$

establishing the lemma.

Now we are going to prove the theorem. Let S be an extremal generating system of an abelian group G. We have to show that for every finite subsystem $a_1, \ldots, a_k$ of S a relation $t_1 a_1 + \cdots + t_k a_k = 0$ implies $t_1 a_1 = \cdots = t_k a_k = 0$. Assume this is not true. Then we have

(1) $$t_1 a_1 + \cdots + t_k a_k = 0, \qquad t_i a_i \neq 0 \qquad (i = 1, \ldots, k)$$

for some subsystem $a_1, \ldots, a_k$ of S. Let

(2) $$\min_{1 \leq i \leq k} O(a_i) = O(a_1).$$

Furthermore we may choose t_1 such that

(3) $$0 < t_1 < O(a_1) \qquad \text{if} \quad O(a_1) < \infty.$$

If $(t_1, \dots, t_k) = t$, $t_i = tr_i$, then $(r_1, \dots, r_k) = 1$ and hence, by the lemma, $\{a_1, \dots, a_k\} = \{b_1, \dots, b_k\}$ with

$$b_1 = r_1 a_1 + \cdots + r_k a_k.$$

Then we have by (1)

$$tb_1 = 0. \tag{4}$$

Thus, according to (3),

$$O(b_1) \leq t \leq t_1 < O(a_1)$$

if $O(a_1) < \infty$, while if $O(a_1) = \infty$ then

$$O(b_1) < O(a_1) \tag{5}$$

immediately follows from (4). However (5) contradicts the extremal property of the system S. This completes the proof.

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