A note on means of entire functions.

By S. M. SHAH in Cambridge (England).

1. Let $f(z)$ be an entire function of order ϱ and lower order λ and let

$$\varrho_1 = \overline{\lim_{r\to\infty}} \frac{\log n(r)}{\log r}$$

$$\lambda_1 = \underline{\lim_{r\to\infty}} \frac{\log n(r)}{\log r}$$

where $n(r)$ denotes the number of zeros of $f(z)$ in $|z| \leqq r$. Let $G(r)$ and $g(r)$ denote the geometric means of $|f(z)|$ on the circumference $|z| = r$ and the circle $|z| \leqq r$ respectively. Then for $r > 0$,

$$G(r) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\vartheta})|\,d\vartheta\right\},$$

$$g(r) = \exp\left\{\frac{1}{\pi r^2}\int_0^r\int_0^{2\pi} \log|f(\varrho e^{i\vartheta})|\,\varrho\, d\varrho\, d\vartheta\right\}.$$

We have $G(r) \leqq \exp(T(r))$, $g(r) \leqq \exp\left\{\frac{2}{r^2}\int_0^r T(\varrho)\varrho\, d\varrho\right\}$. If $n(r) = 0$ then $G(r) = |f(0)| = g(r)$.

Suppose now that $n(r) > 0$ for $r > r_0$ and write

$$K = \overline{\lim_{r\to\infty}} \left(\frac{g(r)}{G(r)}\right)^{1/n(r)}$$

$$k = \underline{\lim_{r\to\infty}} \left(\frac{g(r)}{G(r)}\right)^{1/n(r)}.$$

It is known that[1]) (I) if $0 < \varrho_1 < \infty$, then

(1) $$k \leqq \exp\left(\frac{-1}{2+\varrho_1}\right) \leqq K$$

[1]) See [1], II. p. 10, problems 64, 65, 66,

and that (II) if $n(r) \sim r^{\varrho_1} l(r)$ or $r^{\varrho_1}/l(r)$ where $l(r)$ is a monotone increasing function such that $l(2r) \sim l(r)$ then $k = K$. I prove here

Theorem 1. *If* $\lim_{r\to\infty} n(r) > 0$, *then*

$$\exp\left(-\frac{1}{2}\right) \leqq k \leqq \exp\left(\frac{-1}{2+\lambda_1}\right) \leqq \exp\left(\frac{-1}{2+\varrho_1}\right) \leqq K \leqq 1. \tag{2}$$

Theorem 2. *Let $\Phi(x)$ be positive and continuous for $x > 1$ and satisfy the condition $\Phi(ax) \sim \Phi(x)$, as $x \to \infty$, for every fixed positive a. If $n(r) \sim r^{\varrho_1}\Phi(r)$ then $k = K$.*

2. Let $\tilde{M}(r)$ and $m(r)$ denote the arithmetic means of $|f(z)^2|$ on $|z| = r$ and $|z| \leqq r$ respectively. Let

$$L = \overline{\lim_{r\to\infty}} \left(\frac{m(r)}{\tilde{M}(r)}\right)^{1/\log r}$$

$$l = \underline{\lim_{r\to\infty}} \left(\frac{m(r)}{\tilde{M}(r)}\right)^{1/\log r}.$$

It is known that [1]) when $\varrho < \infty$, $l = e^{-\varrho}$. I prove here

Theorem 3. *For* $0 \leqq \lambda \leqq \infty$, $L = e^{-\lambda}$. *When* $\varrho = \infty$, $l = 0$.

3. *Proof of Theorem* 1. Let $(z_n)_1^\infty$ denote the zeros of $f(z)$, $|z_n| = r_n$ and suppose that $0 = r_1 = \cdots = r_q < r_{q+1} \leqq r_{q+2} \cdots (q \geqq 0)$.

(I) Let $n > q+1, r_n \leqq r < r_{n+1}, f(z) = z^q F(z)$. Then $G(r,f) = r^q G(r,F)$; $g(r,f) = r^q e^{-q/2} g(r,F)$,

$$\left(\frac{g(r)}{G(r)}\right)^{1/n(r)} = \exp\left(-\frac{1}{2} + \frac{1}{2n(r)} \sum_{r_\nu \leqq r} \left(\frac{r_\nu}{r}\right)^2\right). \tag{3}$$

Further

$$\sum_{r_\nu \leqq r} (r_\nu)^2 = \int_0^r x^2\, dn(x) = r^2 n(r) - 2\int_0^r x n(x)\, dx.$$

Hence

$$\left(\frac{g(r)}{G(r)}\right)^{1/n(r)} = \exp\left(-\frac{1}{r^2 n(r)} \int_0^r x n(x)\, dx\right) \tag{4}$$

and so $k \geqq \exp\left(-\frac{1}{2}\right), K \leqq 1$.

(II) We now prove $k \leqq \exp\left(\frac{-1}{2+\lambda_1}\right)$. We suppose $\lambda_1 < \infty$, for if $\lambda_1 = \infty$ then this follows from the relation $g(r) \leqq G(r)$. We also suppose that $n(r) \to \infty$, for if $n(r)=O(1)$, it would follow from (3) that $k=K=\exp\left(-\frac{1}{2}\right)$. We have

$$\lambda_1=\varlimsup_{n\to\infty}\frac{\log n}{\log r_n}.$$

Hence if $\beta>\lambda_1$, $\varliminf_{n\to\infty} n/r_n^\beta=0$. Hence we can find a sequence of integers n_p $(p=1,2,\ldots)$ such that [2])

$$\frac{N}{r_N^\beta}<\frac{\mu}{r_\mu^\beta},\ \mu=q+1,q+2,\ldots,N-1;\ N=n_p.$$

Hence $r_N>r_{N-1}$. Choose r such that $r_{N-1}<r<r_N$ and

$$\frac{N}{r^\beta}\leqq\frac{\mu}{r_\mu^\beta}\ \text{ for }\ \mu=q+1,q+2,\ldots,N-1.$$

Then

$$\left(\frac{r_\mu}{r}\right)^2\leqq\left(\frac{\mu}{N}\right)^{2/\beta}\ \text{ for }\ \mu=1,2,\ldots,N-1.$$

$$\sum_1^{N-1}\left(\frac{r_\mu}{r}\right)^2\leqq\frac{(N-1)^\beta}{2+\beta}+o(N).$$

Hence for these values of r we have from (3)

$$\left(\frac{g(r)}{G(r)}\right)^{1,n(r)}\leqq\exp\left[-\frac{1}{2}+\frac{1}{2(N-1)}\left(\frac{(N-1)\beta}{2+\beta}+o(N)\right)\right]$$

and our statement follows.

(III) We now prove that when $\varrho_1=\infty, K=1$. Since $\varrho_1=\infty$, we have for any (arbitrary large) positive constant H, $\varlimsup_{x\to\infty} n(x)/x^H=\infty$. Hence we can choose a sequence of points R_n $(n=1,2,\ldots)$ such that

$$\frac{n(x)}{x^H}\leqq\frac{n(R_n)}{R_n^H}\ \text{ for }\ r_{q+1}\leqq x\leqq R_n.$$

Set $R_n=R$. Then

$$\frac{1}{R^2n(R)}\int_0^R xn(x)\,dx<\left(O(1)+\int_{r_{q+1}}^R\frac{x^{H+1}n(R)\,dx}{R^H}\right)\frac{1}{R^2n(R)}<o(1)+\frac{1}{H+2}.$$

Hence

$$\varliminf_{r\to\infty}\frac{1}{r^2n(r)}\int_0^r xn(x)\,dx=0$$

and from (4) we get $K=1$. This completes the proof of Theorem 1.

[2]) See [1], I, p. 18, problem 107.

4. We now show that (2) is "best possible".

Example 1. Given ϱ, $0 < \varrho < \infty$, let

$$a_n = 2^{2^{n-1}} (n \geqq 1),\ a_0 = 0,\ p = 1 + [\varrho],$$
$$\xi_n = [a_n^{\varrho/p}] - [a_{n-1}^{\varrho/p}],\ f(z) = \prod_{n=1}^{\infty} \left\{ 1 + \frac{z^p}{a_n} \right\}^{\xi_n}.$$

Then $f(z)$ is a canonical product of order ϱ and $\lambda_1 = \varrho/2$. For this function $k = \exp\left(-\frac{1}{2}\right)$, $K = 1$.

Example 2. Given λ_1 and ϱ_1, $0 < \lambda_1 < \varrho_1 < \infty$, let $p = 1 + [\varrho_1]$, $x_1 = 3$, $x_{n+1} = (x_n!!)^n$ $(n \geqq 1)$, $a_1 = a_2 = 1$,

$$\begin{aligned} a_m &= m^{1/\varrho_1} && \text{for } x_n \leqq m \leqq x_n! \\ &= m^{1/\lambda_1} && \text{for } x_n! < m < x_n!! = z_n \text{ (say)} \\ &= Am + B && \text{for } z_n \leqq m \leqq x_{n+1}, \end{aligned}$$

where A and B are chosen such that

$$a_{x_{n+1}} = (x_{n+1})^{1/\varrho_1},\ a_{z_n} = (z_n)^{1/\lambda_1}; \qquad (n = 1, 2, 3, \cdots)$$

Let

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \left(\frac{z}{a_n} \right)^p \right).$$

Then $f(z)$ is a canonical product of order ϱ_1 for which

$$\varliminf_{r \to \infty} \frac{\log n(r)}{\log r} = \lambda_1,\ k = \exp\left(\frac{-1}{2 + \lambda_1} \right),\ K = \exp\left(\frac{-1}{2 + \varrho_1} \right).$$

5. *Proof of Theorem* 2. We have [3])

$$\int_0^r x n(x)\, dx \sim \int_1^r x^{1+\varrho_1} \Phi(x)\, dx \sim \frac{r^{2+\varrho_1}}{2 + \varrho_1} \Phi(r) \sim \frac{r^2}{2 + \varrho_1} n(r).$$

Hence from (4), the theorem follows.

6. *Proof of Theorem* 3. If $f(z) = \sum_0^{\infty} a_n z^n$ and $M(r) = \underset{|z|=r}{\operatorname{Max}} |f(z)|$ then we have

$$\tilde{M}(r) = \sum_0^{\infty} |a_n|^2 r^{2n},\ m(r) = \sum_{n=0}^{\infty} \frac{|a_n|^2 r^{2n}}{n+1}.$$

Let

$$H(z) = \sum_0^{\infty} \frac{|a_n|^2 z^{2n+2}}{n+1};\ \mu(r) = \text{maximum term of } f(z).$$

Then

$$\frac{r^2 \{\mu(r)\}^2}{\nu(r) + 1} < H(r) < r^2 \{M(r)\}^2.$$

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[3]) See [2], p. 54, Lemma 5.

Hence $\varrho \equiv \varrho(f) = \varrho(H)$ and $\lambda \equiv \lambda(f) = \lambda(H)$, $(0 \leqq \varrho \leqq \infty)$. Further

$$\frac{\widetilde{M}(r)}{m(r)} = \frac{rH'(r)}{2H(r)}$$

and [4])

$$\overline{\lim_{r\to\infty}} \frac{1}{\log r} \log \frac{rH'(r)}{H(r)} = \varrho,$$

$$\underline{\lim_{r\to\infty}} \frac{1}{\log r} \log \frac{rH'(r)}{H(r)} = \lambda.$$

Hence Theorem 3 follows.

References.

[1] G. Pólya und G. Szegő, Aufgaben und Lehrsatze aus der Analysis I, II. Berlin, 1925.
[2] G. H. Hardy and W. W. Rogosinki, Notes on Fourier series. III. *Quarterly J. Math.* (Oxford series) **16** (1945), 49—58.
[3] S. M. Shah, A note on the derivatives of integral functions. *Bull. Amer. Math. Soc.,* **53** (1947), 1156—1163.

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[4]) See [3], pp. 1156—59.