Points of continuity of semi-continuous functions.

By M. K. FORT, Jr. in Urbana, Illinois, U. S. A.

1. Introduction.

Let X be a metric space with distance function d, and let 2^X be the set of all non-empty compact subsets of X. If $A \in 2^X$ and $\varepsilon > 0$, then we define

$$U(\varepsilon, A) = \{x | x \in X \text{ and } d(x, a) < \varepsilon \text{ for some } a \in A\}.$$

We let f be a function on a topological space T into 2^X. The function f is *upper semi-continuous [lower semi-continuous]* at $p \in T$ if and only if corresponding to each $\varepsilon > 0$ there exists a neighborhood V of p such that $f(q) \subset U(\varepsilon, f(p))$ $[f(p) \subset U(\varepsilon, f(q))]$ for all $q \in V$. f is *continuous* at p if and only if f is both upper and lower semi-continuous at p. We prove that if f is either upper semi-continuous on T or lower semi-continuous on T, then the points of T at which f is continuous form a residual set. The first theorem of this type seems to have been proved by L. S. HILL [3]. Extensions of HILL's theorem and similar results have been obtained by KURATOWSKI [4], CHOQUET [1] and FORT [2]. The theorems proved in these papers, however, all contain hypotheses which require X to be either compact or separable. We obtain theorems for non-separable spaces X by making use of functions M and N (defined below) which act somewhat as measure functions on 2^X.

2. Two theorems concerning points of continuity.

Let $A \in 2^X$ and suppose $\varepsilon > 0$. We define $M(A, \varepsilon)$ to be the largest positive integer m for which there exist m points $x_1, \ldots, x_m$ in A with $d(x_j, x_k) > \varepsilon$ for $j \neq k$. We define $N(A, \varepsilon)$ to be the smallest integer n such that A can be covered by n ε-neighborhoods of points of X. The compactness of A insures that $M(A, \varepsilon)$ and $N(A, \varepsilon)$ are properly defined.

Theorem 1. *If f is lower semi-continuous on T, then f is continuous at point of a residual set.*

Proof. If n is a positive integer and $\varepsilon > 0$, we define $B(n, \varepsilon)$ to be the set of all points $p \in T$ for which $M(f(p), \varepsilon) \leqq n$, and for which $0 < \varepsilon' < 3\varepsilon$

and V a neighborhood of p implies that there exists $q \in V$ such that $f(q) \overline{\subset} U(\varepsilon', f(p))$. We shall prove that each set $B(n, \varepsilon)$ is nowhere dense.

We first prove that $B(n, \varepsilon)$ is closed. Since f is lower semi-continuous, it is easy to verify that $M(f(q), \varepsilon) \geqq M(f(p), \varepsilon)$ for all q sufficiently close to p. Let $r \in \overline{B(n, \varepsilon)}$. It follows from the preceding remark that $M(f(r), \varepsilon) \leqq n$. Now let us assume that $0 < \varepsilon' < 3\varepsilon$ and that V is a neighborhood of r. We choose ε'' such that $\varepsilon' < \varepsilon'' < 3\varepsilon$. It follows from the lower semi-continuity of f that $U(\varepsilon', f(r)) \subset U(\varepsilon'', f(p))$ for all p sufficiently close to r. Choose $p \in V \cap B(n, \varepsilon)$ for which $U(\varepsilon', f(r)) \subset U(\varepsilon'', f(p))$. Since $p \in B(n, \varepsilon)$ and V is a neighborhood of p, there exists $q \in V$ for which $f(q) \overline{\subset} U(\varepsilon'', f(p))$. Thus $f(q) \overline{\subset} U(\varepsilon', f(r))$ and we see that $r \in B(n, \varepsilon)$. We have therefore proved that $B(n, \varepsilon)$ is closed.

Next, in order to prove that $B(n, \varepsilon)$ is nowhere dense, we must show that $B(n, \varepsilon)$ cannot contain a non-empty open set. Since $M(f(p), \varepsilon) \leqq n$ for each $p \in B(n, \varepsilon)$, in order to show that $B(n, \varepsilon)$ cannot contain a non-empty open set it is sufficient to prove that for each $p \in B(n, \varepsilon)$ there exist points q arbitrarily close to p for which $M(f(q), \varepsilon) \geqq M(f(p), \varepsilon) + 1$. We let $p \in B(n, \varepsilon)$ and let $m = M(f(p), \varepsilon)$. There exist m points $x_1, \ldots, x_m$ in $f(p)$ such that $d(x_j, x_k) > \varepsilon$ for $j \neq k$. We define $2\delta = \min\{d(x_j, x_k) - \varepsilon \mid j \neq k\}$ and $\eta = \min(\delta, \varepsilon)$. Now let V be a neighborhood of p. There exists $q \in V$ such that $f(p) \subset U(\eta, f(q))$ and $f(q) \overline{\subset} U(2\varepsilon, f(p))$. We choose points $y_1, \ldots, y_m$ in $f(q)$ such that $d(y_k, x_k) < \eta$ for $1 \leqq k \leqq m$ and we choose $y_{m+1} \in f(q) - U(2\varepsilon, f(p))$. It is easy to verify that if j and k are different integers between 1 and $m+1$ then $d(y_j, y_k) > \varepsilon$. Thus $M(f(q), \varepsilon) \geqq m+1$.

We now let B be the union of the sets $B(n, \varepsilon)$ for all positive integers n and positive rational numbers ε. B is of the first category, and hence $T - B$ is a residual set. Our theorem is proved if we can show that f is upper semi-continuous (and hence continuous) at each point of $T - B$.

Let $p \in T - B$ and suppose $\varepsilon > 0$. We choose ε' rational such that $0 < \varepsilon' < \varepsilon/3$ and let $m = M(f(p), \varepsilon')$. Since $p \notin B(n, \varepsilon')$, there exists a positive number $\varepsilon'' < 3\varepsilon'$ and a neighborhood V of p such that $f(q) \subset U(\varepsilon'', f(p))$ for all $q \in V$. Thus $f(q) \subset U(\varepsilon, f(p))$ for all $q \in V$ and we see that f is upper semi-continuous at p.

Theorem 2. *If f is upper semi-continuous on T, then f is continuous at points of a residual set.*

Proof. For each positive number ε we define $C(\varepsilon)$ to be the set of all points $p \in T$ such that $0 < \varepsilon' < 3\varepsilon$ and V a neighborhood of p implies that there exists $q \in V$ such that $f(p) \overline{\subset} U(\varepsilon', f(q))$. We shall prove that each set $C(\varepsilon)$ is nowhere dense.

First we prove that $C(\varepsilon)$ is closed. Let $r \in \overline{C(\varepsilon)}$ and suppose $0 < \varepsilon' < 3\varepsilon$. Choose ε'' such that $\varepsilon' < \varepsilon'' < 3\varepsilon$. Since f is upper semi-continuous, there exists a neighborhood V of r such that if $p \in V$ then $f(p) \subset U(\varepsilon'' - \varepsilon', f(r))$. We now choose $p \in V \cap C(\varepsilon)$. There exists $q \in V$ such that $f(p) \,\overline{\subset}\, U(\varepsilon'', f(q))$. It follows easily that $f(r) \,\overline{\subset}\, U(\varepsilon', f(q))$ and hence $r \in C(\varepsilon)$. Thus $C(\varepsilon)$ is closed.

We must now prove that $C(\varepsilon)$ cannot contain a non-empty open set. Since the numbers $N(f(p), \varepsilon)$ are bounded below by 0, in order to prove that $C(\varepsilon)$ cannot contain a non-empty open set it is sufficient to prove that if $p \in C(\varepsilon)$ then there exist points q arbitrarily close to p for which $N(f(q), \varepsilon) \leqq N(f(p), \varepsilon) - 1$. Let $p \in C(\varepsilon)$ and let $n = N(f(p), \varepsilon)$. Choose a set $W_1, \ldots, W_n$ of ε-neighborhoods which cover $f(p)$. It follows from the upper semi-continuity of f that there exists a neighborhood V of p such that $f(q) \subset \bigcup_{k=1}^{n} W_k$ for all $q \in V$. Since $p \in C(\varepsilon)$, there exists $q \in V$ such that $f(p) \,\overline{\subset}\, U(2\varepsilon, f(q))$. Choose $x \in f(p) - U(2\varepsilon, f(q))$. There exists k such that $x \in W_k$. It is easy to see that no point of $f(q)$ is in W_k. Hence $f(q)$ can be covered by a set of $n-1$ ε-neighborhoods and $N(f(q), \varepsilon) \leqq n-1$.

We now let C be the union of all sets $C(\varepsilon)$ for which ε is rational. C is a set of the first category and hence $T - C$ is a residual set. It is easy to verify that f is lower semi-continuous at each point of $T - C$ and hence f is continuous at each point of a residual set.

Bibliography.

[1] G. Choquet, Convergences. *Annales de L'Université de Grenoble.* **23** (1947), 57—111.
[2] M. K. Fort, Jr., A unified theory of semi-continuity. *Duke Math. J.*, **16** (1949), 237—246.
[3] L. S. Hill, Properties of certain aggregate functions. *Amer. J. Math.* **49** (1927), 419—432.
[4] C. Kuratowski, Sur les décompositions semi-continues d'espaces métriques compacts. *Fund. Math.* **11** (1928), 169—186.

(Received April 26, 1951.)