Remarks on quasigroups and n-quasigroups.

By DAVID ELLIS in Florida and ROY UTZ in Missouri, U. S. A.

1. Introduction.

In this note we exhibit some remarks pertaining to previous studies of quasigroups by other authors. We also consider a new (as far as we can ascertain) relation among quasigroups; namely, the right and left associates of a given quasigroup, and a new type of system which is, in a sense, a generalization of the notion of quasigroup.

We define an n-quasigroup (n a positive itneger) as a groupoid Q closed under the operation ab which contains at least n distinct elements and in which the two systems of equations:

$$\text{(T)} \qquad \begin{matrix} a_i x = b_i \\ y a_i = b_i \end{matrix} \qquad (i = 1, 2, \ldots, n).$$

heve each a unique solution, x and y, in Q whenever $\{a_i\}$ and $\{b_i\}$ are n-tuples each consisting of n pairwise distinct elements of Q.

The 1-quasigroups coincide with the usual quasigroups.

The n-quasigroup is the algebraic correspondent (metroid) of a generalized semimetric ground space in which each n distinct points form a complete metric base. [1])

2. Right identity graphs of quasigroups.

A. C. CHOUDHURY [2]) has exploited the graph of the right identity ordering relation of a quasigroup. We list a few elementary observations on these. Of course, similar remarks apply to left identity graphs. In this section, Q denotes a quasigroup and G denotes the right identity graph of Q.

Remark 1. If Q is finite and if $xs = x$ is solvable in Q for x then x is a single valued function of s and every component of G is a cycle or a

[1]) DAVID ELLIS, Notes on abstract distance geometry. I. The algebraic description of ground spaces. Unpublished. Abstracted for Amer. Math. Soc. Meeting, Christmas, 1950.

[2]) A. C. CHOUDHURY, Quasigroups and nonassociative systems. I., II. *Bull. Calcutta Math. Soc.* **40** (1948), 183—194, and **41** (1949), 211—219.

single point (an isolated point of G corresponds to an idempotent element of Q but not, in general, conversely).

Remark 2. If $xs = x$ is uniquely solvable in Q (not necessarily finite) then each component of G is a topological line, a cycle, or a single point.

Remark 3. The number of idempotent elements of Q does not exceed the number of components of G.

Remark 4. Q is determined up to isomorphism by G if and only if the order of Q does not exceed three.

3. Remarks on isotopy.

Let A and B be groupoids under $a_1 * a_2$ and $b_1 b_2$ respectively. A and B are called isotopic if there are three one-to-one mappings, α, β, γ of A onto B so that for each a_1, a_2 in A we have: $\alpha(a_1 * a_2) = \beta(a_1)\gamma(a_2)$. Two groupoid A and B as above which have the same point set are principal isotopes if they are isotopic as above where α is the identity mapping. ALBERT[3]) has shown that if A and B are isotopic then A is isomorphic to a principal isotope of B. Isotopy is an equivalence relation in any class of groupoids and principal isotopy is an equivalence relation in any class of groupoids having the same point set. Isotopy is a relevant equivalence relation in the study of quasigroups among groupoids since the quasigroup property is invariant under isotopy[4]). The following example shows, however, that isotopy preserves, in general, neither associativity nor commutativity so that it is not a relevant equivalence relation in the study of groups and abelian groups among quasigroups.

Example. Let S be the set of reals which are positive. S forms a non-associative, non-commutative right loop under $a * b$ where $a * b = ab^2$ and xy denotes ordinary multiplication. S forms an abelian group under ab. Yet with $\alpha(x) = x$ and $\beta(x) = x^{\frac{1}{2}}$ we have principal isotopes.

We now use the following two results of ALBERT[3]):

1. Every quasigroup is isotopic to a loop.

2. A loop is isotopic to a group if and only if it is isomorphic to the group and is, hence, itself a group.

We have:

---

[3]) A. A. ALBERT, Quasigroups I. and II. *Trans. Amer. Math. Soc.* **54** (1943), 507—519, and **55** (1944), 401—419.

[4]) R. H. BRUCK, Some results in the theory of quasigroups *Trans. Amer. Math. Soc.* **55** (1944), 19—52.

Theorem 1. *If the groupoids admissable from some universe of discourse are divided into isotopy classes and if* $\mathbf{G}(\mathbf{C}), \mathbf{L}(\mathbf{C}), \mathbf{S}(\mathbf{C})$, *and* $\mathbf{Q}(\mathbf{C})$ *denote the subclasses consisting of all groups, loops, semigroups, and quasigroups, respectively, in such an isotopy class* $\mathbf{C}$ *then*:

a) $\mathbf{Q}(\mathbf{C}) = 0$ *or* $\mathbf{Q}(\mathbf{C}) = \mathbf{C}$.

b) $\mathbf{G}(\mathbf{C}) = 0$ *or* $\mathbf{G}(\mathbf{C}) = \mathbf{L}(\mathbf{C})$.

c) *If* $\mathbf{Q}(\mathbf{C}) \neq 0$ *then* $\mathbf{L}(\mathbf{C}) \neq 0$.

d) *If* $\mathbf{Q}(\mathbf{C}) \neq 0$ *and* $\mathbf{S}(\mathbf{C}) \neq 0$ *then* $\mathbf{S}(\mathbf{C}) = \mathbf{L}(\mathbf{C}) = \mathbf{G}(\mathbf{C})$.

We offer the following conjecture on the characterization of isotopy among equivalence relations on quasigroups. The necessity part follows from theorem 1.

Conjecture. *Let the class of all quasigroups admissable from some universe of discourse be divided into disjoint classes. In order that the equivalence relation defined by this division shall be equivalent to isotopy it is necessary and sufficient that the following conditions subsist:*

a) The given classification in refinable by isomorphism (that is, isomorphic quasigroups lie in the same class).

b) $\mathbf{C} \neq 0$ *implies* $\mathbf{L}(\mathbf{C}) \neq 0$ *for each class* $\mathbf{C}$.

c) $\mathbf{S}(\mathbf{C}) \neq 0$ *implies* $\mathbf{S}(\mathbf{C}) = \mathbf{L}(\mathbf{C}) = \mathbf{G}(\mathbf{C})$ *for each class* $\mathbf{C}$.

d) All members of a given class have the same cardinal (for any given class).

4. Associated quasigroups.

Let Q be a quasigroup under ab. Define arb to be the x defined by $ax = b$ and alb to be the y defined by $yb = a$. The point set of Q forms quasigroups Q_r and Q_l under arb and alb, respectively, called the right and left associates of Q. This notion of association among three quasigroups arose in a discussion between Gaines Lang and one of the writers.

We say that Q has peroid 2 with respect to an element c if $a^2 = c$ for all a in Q.

We find:

Remark 1. $(Q_r)_r = Q = (Q_l)_l$.

As a consequence of this remark and the obvious fact that Q determines its associates up to isomorphism we have:

Theorem 2. *Each of the triple* Q, Q_r, Q_l *determines the others up to isomorphism.*

Remark 2. Q is a right loop (left loop) if and only if Q_r (Q_l) is of period 2 with respect to some element.

Remark 3. Q is a right loop (left loop) if and only if Q_l (Q_r) is a right loop (left loop).

Combining Remarks 1, 2, and 3 we have:

Theorem 3. *If Q is a loop then Q_r is a left loop and Q_l is a right loop, and conversely. Q_r and Q_l are both loops if and only if Q is a loop of period two with respect to its identity.*

Since, by definition, $b l a = x$ if $xa = b$ and $a r b = x$ if $ax = b$ we have:

Theorem 4. *Q is commutative if and only if Q_r and Q_l are skew-isomorphic under the identity mapping.*

Remark 4. Q, Q_r, and Q_l are comcomitantly idempotent. That is, an element idempotent in any one of the three operations of these quasigroups is idempotent in all of them.

5. Existence of n-quasigroups and invariance under isotopy.

We observe first that any quasigroup of two elements is a 2-quasigroup. We also observe that any n distinct elements of an n-quasigroup imply the existence of $n!$ elements as solutions to equations (T) of the definition of n-quasigroup. These are distinct and each n of them imply the existence of $n!$ more elements distinct from the preceding ones.

A little reflection shows that one may construct an n-quasigroup for any positive integer n from a denumerable set.

Thus we have the

Theorem 5. *The only finite n-quasigroups with $n > 1$ are the two 2-quasigroups of order* 2. *Infinite n-quasigroups exist for any positive integer n.*

We have the following theorem for n-quasigroups. The proof is similar to the proof of the corresponding theorem for quasigroups and is left to the reader.

Theorem 6. *Every isotope of an n-quasigroup is an n-quasigroup.*

Garrison[5]) has shown that the order of a subquasigroup of a finite quasigroup does not necessarily divide the order of the full quasigroup, contrary to the case in group theory. We offer the somewhat related conjecture:

Conjecture. *If M is an n-quasigroup and N is a subgroupoid of M wich forms an m-quasigroup under the operation of M with $m < n$ then m divides n.*

6. Associative n-quasigroups.

A quasigroup wich is also a semigroup (is associative) is, of course, a group. We now find a more startling result for n-quasigroups whith $n > 1$. It is well known that an associative groupoid in which $ax = b$ and $ya = b$ have

[5]) George N. Garrison, Quasigroups, *Annals of Math.* **41** (1940), 474—487.

solutions is a group. Thus every n-quasigroup which is also a semigroup is a group.

Lemma. *If M is an associative n-quasigroup with $n > 1$ and a and b are non-idempotent elements of M then $ab = ba$.*

Proof. By hypothesis, $a \neq a^2$ and $b \neq b^2$. If $a = b$, the result is immediate. Otherwise, there are x and y so that $ax = b$ and $a^2x = b^2$, $ya = b$ and $ya^2 = b^2$. Then $ab = a(ax) = (aa)x = a^2x = b^2 = ya^2 = y(aa) = (ya)a = ba$.

Since the only idempotent element of a group is the identity we have

Theorem 7. *Every n-quasigroup which is also a semigroup is a group. If $n > 1$, the group is abelian.*

(Received May 7, 1951.)