Publ. Math. Debrecen
47 / 3-4 (1995), 237–248

On completely integrable systems

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The objective of this paper is to give a geometric formulation, using contact theory, for differential systems of order k and dimension n over a differentiable manifold. In the case $k = 1$ we obtain the classical Frobenius theorem.

Let (M, N, π) be a fibred manifold and $J^k M$ the k-jets bundle of cross-sections of (M, N, π). Two functions defined on a neighborhood of $X \in J^k M$ are and identified, and the equivalence class is called a germ of functions at X. The set of all germs at X is denoted by $\Omega_X J^k M$. Given an open set $\mathbf{U} \subset J^k M$, let $\Omega\mathbf{U} = \bigcap\{\Omega_X,\ X \in \mathbf{U}\}$.

A System of Partial Differential Equations (P.D.E.) of order k in (M, N, π) is defined as an open set $\mathbf{U}$ in $J^k M$, together with a locally, finitely generated, subsheaf of ideals Φ of $\Omega\mathbf{U}$ [3]. The set $\mathbf{U}$ is called the domain of the S.P.D.

A cross-section f of (M, N, π) is said to be a solution of the equation $\Phi = 0$ (or a solution of Φ, for simplicity) if and only if, for any $x \in \operatorname{Dom} f$, the jet $j^k_x f$ is in the domain of Φ and $F(j^k_x f) = 0$, for any F belonging to Φ. If $j^k_{x_0} f = X_0$ we say that f is a solution at X_0.

An integral jet X of a S.P.D. Φ, with domain $\mathbf{U}$, and order k is a k-jet $X \in \mathbf{U}$ such that $F(X) = 0$, for all F belonging to Φx. The set of all the integral jets of Φ is denoted by $J\Phi$. A cross-section f of (M, N, π) is then a solution of Φ if and only if $j^k_x f \in J\Phi$, for all $x \in \operatorname{Dom} f$.

A system Φ of order k defined on a fibred manifold (M, N, π) is said to be completely integrable at $X_0 \in J\Phi$ if there exists a solution f of Φ at X_0.

Denote by $C^{k,n} M$, the contact bundle of order k of n-dimensional submanifolds in M, and by $C^k_x S$ the contact element of order k at $x \in S \subset M$ [2].

Given an imbedded submanifold $W \subset C^{k,n}M$, we will define an associated system of partial differential equations Φ in (V, U, π), where $V \subset M$ is an open subset fibered over the open set $U \subset \Re^n$. We will give sufficient conditions on W for the complete integrability of Φ. Moreover, if $X_0 \in W$ and $S \subset M$ is an n-submanifold such that $C^k_x S \in W$, for all $x \in S$ and $C^k_{x_0} S = X_0$ then, we can obtain a solution of Φ using a local parametrization of S in a neigborhood of x_0.

This allows us to give a geometrical interpretation of a completely integrable system of order k, using contact theory.

For $k = 1$, the submanifold W defines a integrable distribution on M, yielding the classical Frobenius Theorem.

1. Jet theory and completely integrable systems

Let (M, N, π) be a fibred manifold, and f, g two cross-sections whose domains contain $x_0 \in N$. Let k be an integer, $k \geq 0$. We say that f and g are k-equivalent at x_0 if the following condition is satisfied:

For any fibred chart (x, y), where the domain of (x) contains x_0, and for any partial derivative ∂^l in (x) of order $l \leq k$ we have: $\partial^l f^j(x_0) = \partial^l g^j(x_0)$, where $f^j = y^j \circ f$ (resp. $g^j = y^j \circ g$) is the expression of f (resp. of g) in terms of (x, y).

If f is a cross-section whose domain contains x_0, the equivalence class containing f is called k-jet of f at x_0 and will be denoted by $j^k_{x_0} f$. Denote by $J^k_x(M, N, \pi)$ the set of all k-jets at x of cross-sections of (M, N, π), and by $J^k(M, N, \pi)$ the set of all k-jets of cross-sections of (M, N, π). We shall write $J^k_x M$ (resp. $J^k M$) when there is no possibility of confusion.

If $X = j^k_x f$, we set $\alpha(X) = x$ and $\beta(x) = f(x)$, thus α (resp. β) is a map of $J^k M$ into N (resp. N), which is called the source map (resp. target map).

Denote by $I_l = (i_1, \cdots, i_l)$ an ordered l-uple of integers $1, \cdots, n$, with $n = \dim N$.

Let (x_i, y^j) be a fibred chart defined on $U \subset N$. If $X = j^l_{x_0} f$ is in $\beta^{-1}(U)$ and $f^j(x)$ is the expression of f, then $(x, y, p^j_{I^l}(X))$, $1 \leq l \leq k$, will be called the chart of $J^k M$ associated with (x, y), where

$$p^j_{I^l}(X) = \frac{\partial^l}{\partial x_{i_1} \cdots \partial x_{i_l}} f^j(x_0).$$

We consider the manifold structure on $J^k M$ given by these charts.

We have that $(J^k M, N, \alpha)$ and $(J^k M, N, \beta)$ are fibred manifolds.

For $l \leq k$, consider the map

$$\pi_l^k : j_x^k f \in J^k M \longmapsto j_x^l f \in J^l M,$$

then $(J^k M, J^l M, \pi_l^k)$ is also a fibred manifold [1].

Two functions defined on a neighborhood of $w \in M$ are identified if they coincide in some neighborhood of w. The equivalence class is called germ of functions at w. The germ of a function ϕ at w will be denoted by $[\phi]_w$. The set of germs of functions at w is a ring, and will be denoted by $\Omega_w M$.

We set $\Omega M = \bigcup\{\Omega_w M;\ w \in M\}\cdot$ If Φ is a subset of ΩM, we set $\Phi_w = \Phi \cap \Omega_w M$.

A function ϕ defined on an open set $U \subset M$ is said to be in Φ (denoted $\phi \in \Phi$) if $[\phi]_w \in \Phi$, for any $w \in U$.

The set Φ is called a subsheaf of ΩM if the following two conditions are satisfied for any $w \in M$:

1) Φ is not empty,

2) For any $a \in \Phi_w$ we can find a function ϕ defined on a neighborhood of w such that $a = [\phi]_w$ and $\phi \in \Phi$.

If U is an open subset of M and Φ is a subsheaf, $\bigcup\{\phi_x;\ w \in U\}$ is a subsheaf of ΩU and is called the restriction of Φ to U. A subsheaf Φ of ΩM is called a subsheaf of ideals when Φ_w is an ideal of $\Omega_w M$ for any $w \in M$.

Let $F_1, \ldots, F_s$ be a finite set of functions defined on an open set $U \in M$, and Φ_w, $w \in U$, the ideal of $\Omega_w M$ generated by $[F_1]_w, \ldots, [F_s]_w$. Then $\Phi = \bigcup\{\phi_w;\ w \in U\}$ is a subsheaf of ideals of ΩU which is said to be generated by ΩU.

A subsheaf of ideals is said to be locally finitely generated when its restiction to an open neighborhood of each point is finitely generated.

A *system of partial differential equations* (P.D.E.) of order k in (M, N, π) is an open set $\mathbf{U} \in J^k M$, together with a locally finitely generated subsheaf of ideals of $\Omega\mathbf{U}$. The set $\mathbf{U}$ is called the domain of the P.D.E. .

An *integral jet of* a P.D.E. Φ, with domain $\mathbf{U}$, of order k, is a k-jet $X \in \mathbf{U}$ such that $F(X) = 0$, for all $F \in \Phi$. We denote by $J\Phi$ the set of integral jets of Φ.

A *a solution of* Φ is a cross-section f of (M, N, π), defined over an open set $U \subset \alpha(\mathbf{U})$ such that, for any $x \in U$, the jet $j_x^k f \in \mathbf{U}$. If $X_0 \in \mathbf{U}$ and $j_{x_0}^k f = X_0$, with $x_0 \in U$, we say that f is a solution of Φ at x_0.

Let $F_1, \ldots, F_s$ be a finite set of functions on an open set $\mathbf{U} \in J^k M$. Then the subsheaf Φ of ideals of $\Omega\mathbf{U}$ generated by $F_1, \ldots, F_s$ is a S.P.D. of order k.

Assume that $\mathbf{U}$ is contained in the domain of the chart $(x, y, p^j_{I^l})$ associated with a fibred chart (x, y). Then a cross-section $f = (x, f^j)$ is a solution of the equation $\Phi = 0$ if $f^j(x)$ is a solution of the system of partial differential equations

$$F_r(j^k_x f) = 0, \quad 1 \le r \le s.$$

The system Φ is *complete* at $X \in \mathbf{U}$ if for any function F, defined on an open set $\mathbf{V} \subset U$, which vanishes on $J\Phi \cap \mathbf{V}$, the restriction of F to an open neighborhood of X belongs to Φ.

Let F be a function defined on an open set $\mathbf{U} \subset J^k M$ and θ a vector field on $\alpha(\mathbf{U}) \subset N$. *The formal derivative of* F with respect to θ, denoted by $\partial^{\#}_\theta F$, is a function on $(\pi^{k+1}_k)^{-1}(\mathbf{U})$ defined as follows:

$$\partial^{\#}_\theta F : j^{k+1}_{x_0} f \longmapsto \theta_{x_0}(F \circ j^k f),$$

where $j^k f : \mathrm{Dom}\, f \longrightarrow J^k M$ is defined by $x \longmapsto j^k_x f$. This definition is independent of the choice of representative f [3].

Let Φ be a P.D.E. with domain $\mathbf{U} \subset J^k M$. For any $X^{k+1} \in (\pi^{k+1}_k)^{-1}(\mathbf{U}) = \mathbf{U}'$, let $(p\Phi)_{X^{k+1}}$ be the ideal of germs at X^{k+1} generated by

$$\{F \circ \pi^{k+1}_k,\ \partial^{\#}_\theta F; \quad F \in \Phi_{X^k},\ \theta \in \chi(\alpha(\mathbf{U}))\}.$$

The subsheaf of ideals of $\Omega\mathbf{U}'$ generated by $(p\Phi)_{X^{K+1}}$, with $X^{k+1} \in \mathbf{U}'$, is called *the prolongation of* Φ and denoted by $P\Phi$.

Given $X^k \in J^k M$ denote by $Q_{X^k} J^k M$ (or Q_{X^k} when there is no possibility of confusion) the set,

$$Q_{X^k} = \text{kernel } \{d\pi^k_{k-1} : T_{X^k} J^k M \longrightarrow T_{X^{K-1}} J^{k-1} M\}$$

and $C_{X^k}\Phi$ the vector subspace defined by

$$C_{X^k}\Phi = \{v \in T_{X^k} Q_{X^k} : v(F) = 0;\ F \in \Phi_{X^k}\}.$$

This subspace is called *the Symbol* of Φ at X^k.

A system of partial differential equations Φ is said to be *completely integrable at* $X \in J\Phi$ if the following conditions are satisfied:

1) $C_X \Phi = 0$,
2) The image of $J(P\Phi)$ by π^{k+1}_k is a neighborhood of X in $J\Phi$,
3) Φ is complete at X.

Theorem 1. *Assume that a P.D.E. Φ of order k is completely integrable at $X \in J\Phi$. Then there is a solution f of Φ at X; moreover, the germ of f at $\alpha(X)$ is uniquely determined.*

Proof. (see [3]).

A system of partial differential equations Φ, with domain $\mathbf{U}$, is *regular* at $X \in \Phi$ if:

i) $J\Phi$ is a submanifold on a neighborhood of X,

ii) there exist functions $\{F_1, \cdots, F_s, \quad F_i \in \Phi\}$, where $s + \dim J\Phi = \dim \mathbf{U}$, such that, $\{dF_1, \cdots, dF_s\}$ are linearly independent at X (as elements in $T^*_X\mathbf{U}$).

Proposition 1. *Let Φ be a regular S.P.D. defined on an open set $\mathbf{U} \subset J^kM$. Suppose that $J\Phi$ is a regular submanifold in J^kM, fibered on $U \subset N$ by α, then*

$$J(P\Phi) = J^{k+1}M \cap J^1(J\Phi, U, \alpha).$$

Proof. Let $X \in J(P\Phi)$ be an integral jet of the prolongation of Φ, defined by $X = j^{k+1}_u f$ and

$$H_X = T_u(j^k f)(T_u U).$$

It is clear that if $Y \in J(P\Phi)$ then,

$$X = Y \Longleftrightarrow H_X = H_Y,$$

then X can be identified with the subspace H_X. Moreover, $X \in J(P\Phi)$ if and only if $H_X \subset T_{\pi^{k+1}_k(X)} J\Phi$, indeed:

$$\begin{aligned} X \in J(P\Phi) &\Leftrightarrow F(X) = 0, \text{ and } \partial^{\#}_{x_i}(F \circ J^k f)|_X = 0, \quad F \in \Phi \\ &\Leftrightarrow \frac{\partial}{\partial x_i}\Big|_u (F \circ j^k f) = 0 \quad \& \quad F(X) = 0, \quad F \in \Phi \\ &\Leftrightarrow T_u(j^k f)\left(\frac{\partial}{\partial x_i}(F)\right) = 0 \quad \& \quad F(X) = 0 \\ &\Leftrightarrow T_u(j^k f)\left(\frac{\partial}{\partial x_i}\right) \in T_X(J\Phi). \end{aligned}$$

On the other hand, a $(k+1)$-jet $X \in P\Phi$ if and only if there exists a section $\sigma : U \longrightarrow J\Phi$ such that $X = j^1_{\alpha(X)}\sigma$. This is clear because, as we have shown, $H_X \in T_{\pi_k^{k+1}(X)} J\Phi$. In consequence, we have:

$$\begin{aligned} X \in J(P\Phi) \Leftrightarrow X = j_u^{k+1} f, &\quad f : U \longrightarrow V \subset M \quad \text{section} \\ \& \quad X = j_u^1 \sigma, &\quad \sigma : U \longrightarrow J\Phi \quad \text{section} \\ \Leftrightarrow X \in J^{k+1} M \cap J^1(J\Phi, U, \alpha). &\quad \square \end{aligned}$$

2. Contact manifolds and jet theory

Let M be a smooth $(n+m)$-dimensional manifold, k an integer, $k \geq 0$, and $\tilde{J}_0^k(\mathfrak{R}^n, M)$ be the k-jets at 0 of maximal rank maps from $\mathfrak{R}^n$ into M.

Given $X_1, X_2 \in \tilde{J}_0^k(\mathfrak{R}^n, M)$ with $\beta(X_1) = \beta(X_2) = x \in M$. We say that X_1 and X_2 are equivalent at $x \in M$ if there exist $Y \in \tilde{J}_0^k(\mathfrak{R}^n, \mathfrak{R}^n)$ such that $X_2 = X_1 \circ Y$.

A class of this equivalence relation is called *the contact element* of order k and dimension n at x, denoted by $[j_x^k f]$. Let $C_x^{k,n} M$ denote the set of all contact elements of order k at x, and $C^{k,n} M$ the set of all k-contact elements of dimension n on M.

Let $S \subset M$ be an imbedded n-dimensional submanifold and f, g, two local parametrizations of S at $x \in M$ over a neighborhood $V \subset M$, with $f(0) = g(0) = x$, then

$$g^{-1}|_{V \cap S} : V \cap S \longrightarrow \mathfrak{R}^n$$

is a local diffeomorphism and

$$h = g^{-1} \circ f : A \subset \mathfrak{R}^n \longrightarrow B \subset \mathfrak{R}^n$$

is a local diffeomorphism such that $j_0^k f = j_0^k g \circ j_0^k h$, thus $[j_0^k f] = [j_0^k g]$.

The equivalence class $[j_0^k f]$ is called the contact element of order k of S at $x \in S$ and denoted $C_x^k S$.

Two imbedded submanifolds S_1, S_2 have contact of order k at $x \in S_1 \cap S_2$ if there exist local parametrizations given by imbeddings

$$f_1, f_2 : U \subset \mathfrak{R}^n \longrightarrow M$$

and a local coordinate system (V, y^j), $1 \leq j \leq m$, about $x \in M$ such that $f_1(0) = f_2(0) = x$ and the partial derivatives at 0 of $y^j \circ f_1$ and $y^j \circ f_2$ are equal up to order k.

Remarks.

1. Cleary $C^1_x S_1 = C^1_x S_2$ if and only if $T_x S_1 = T_x S_2$.

2. Given a map $f : U \subset \Re^n \longrightarrow M$ of maximal rank defined on a neigborhood $U \subset \Re^n$, with $f(u) = x \in M$, and τ_u the traslation in $\Re^n$ such that $\tau_u(0) = u$. Then,

$$j^k_0(f \circ \tau_u) \in \tilde{J}^k_0(\Re^n, M) \quad \& \quad [j^k_0(f \circ \tau_u)] \in C^{k,n}_x M.$$

Let $X \in C^{k,n}M$ and $f : U_f = \mathrm{Dom}\, f \longrightarrow V \subset M$ such that $[j^k_0 f] = X$.

Consider a local coordinate system $(V, \varphi = (x^i, y^j))$, $1 \leq i \leq n$, $1 \leq j \leq m$, such that:

$x^i \circ f = \xi$, the canonical coordinates in $\Re^n$.

Denote by $\pi : \Re^{n+m} \longrightarrow \Re^n$ the canonical projection.

Then (V, U_f, ρ), with $\rho = \pi \circ \varphi$, is a fibred manifold, said to be associated to X, and f is a cross-section.

Define a neighborhood $\mathbf{V}$ of X as:

$$\mathbf{V} = \{Y = C^k_u h(U) \in C^{k,n}V : \ h : U \longrightarrow V \text{ section}, \ u \in U\}.$$

Let

$$\Psi : J^k(V, U, \rho) \longrightarrow \mathbf{V}$$

given by

$$\Psi(j^k_u f) = [j^k_0(f \circ \tau_u)],$$

this map is a bijection and will be denoted by Ψ, when there is no possibility of confusion.

A coordinate neighborhood at $X = [j^k_0 f]$ is given by:

$$(\Psi^{-1}(\mathbf{U}), \eta \circ \Psi^{-1}),$$

where $(\mathbf{U}, \eta)$ is a coordinate neighborhood at $j^k_0 f \in J^k V$.

Consider the manifold structure in $C^{k,n}M$ given by all coordinate neighborhoods defined above.

With this differential structure the natural projections,

$$\rho^k_0 : C^k_x S \in C^{k,n}M \longmapsto x \in M, \quad \& \quad \rho^k_l : C^k_x S \longmapsto C^l_x S \in C^{l,n}M$$

are submersions, and the natural injections,

$$i^k : x \in S \longmapsto C^k_x S \in C^{k,n}M$$

and

$$i^{k+1} : C^{k+1}_x S \in C^{k+1,n}M \longmapsto C^1_{C^k_x S}(C^k S) \in C^{1,n}(C^{k,n}M)$$

are immersions.

Moreover we have the differential map

$$\Psi^{1,k} : J^1(J^kV, \alpha, U) \longrightarrow C^{1,n}(C^{k,n}V)$$

defined by

$$j^1_u(\sigma) \longmapsto C^1_{\sigma(u)}(\Psi(\sigma(U)).$$

3. Differential systems of order k and dimension n

Definitions. By a differential system of order k and dimension n in M we mean an imbedded submanifold $W \subset C^{k,n}M$.

A solution of a differential system W at $X \in W$, is a n-dimensional imbedded submanifold $S \subset M$, with $x = \rho^k_0(X) \in S$, such that $C^kS \subset W$ and $C^k_xS = X$.

Example. Let D be a differentiable distribution of n-planes defined on M. By Remark 1, we can identify a plane $D_x \in D$ with a contact element $D_x \in C^{1,n}M$. Consider the map,

$$\vartheta : x \in M \longmapsto D_x \in C^{1,n}M.$$

If D is a differentiable distribution, then this map is an imbedding, and $\vartheta(M)$ is a differential system of order 1 and dimension n in M.

Moreover, if D is an involutive distribution, then the differential system W has solution.

Definition [5]. The first prolongation of a submanifold $W \subset C^{k,n}M$ is defined as:

$$PW = C^{1,n}W \cap C^{k+1,n}M,$$

where $C^{k+1,n}M$ is identified with its image by $i^{1,k}$ in $C^{1,n}(C^{k,n}M)\cdot$

Theorem. *Let $W \subset C^{k,n}M$ be an imbedded submanifold such that the following conditions are satisfied:*

1) $\rho^k_{k-1} : W \longrightarrow C^{k-1,n}M$, *is a local immersion in a neighborhood of $X \in W$.*

2) $\rho^{k+1}_k : PW \longrightarrow W$ *is a local submersion in a neighborhood of X,*

then there exists a solution $S \subset M$ of the differential system W passing through X. Moreover, if $\tilde{S}$ is another submanifold of W passing through X, then there exists an open set $A \subset M$, $x = \rho^k_0(X) \in A$, such that $S \cap A = \tilde{S} \cap A$.

Proof. Let $f : U \subset \Re^n \longrightarrow V \subset M$, $x \in V$ be an immersion such that $X = [j_0^k f]$, and (V, U, ρ) the fibered manifold associated to X given above.

Let

$$\Psi : \mathbf{U} \subset J^k V \longrightarrow \mathbf{V} \subset C^{k,n} V$$

be the local diffeomorphism defined above.

Let $F_1, \cdots, F_s$ be differentiable functions defined in a neighborhood of X (also written $\mathbf{V}$), such that:

$$\mathbf{V} \cap W = \{X \in \mathbf{V} : F_1 = \cdots = F_s = 0\}.$$

Consider the system of partial differential equations Φ of order k in (V, U, ρ) with domain $\mathbf{U} = \Psi^{-1}(\mathbf{V})$, generated by $\{G_j = F_j \circ \Psi\}$. Let $Y = \Psi^{-1}(X)$.

We will verify that Φ satisfies the hypothesis of Theorem 1.

First, we observe that the integral jets $J\Phi$ are given by $\Psi^{-1}(W)$. Indeed:

$$Y \in J\Phi \Leftrightarrow G_j(Y) = 0 \Leftrightarrow (F_j \circ \Phi)(Y) = 0 \Leftrightarrow \Phi(Y) \in W.$$

Now, by hypothesis $\rho^k_{k-1} : (W \cap \mathbf{V}) \longrightarrow C^{k-1} V$ is an immersion, then

$$\pi^k_{k-1} : (J\Phi \cap \mathbf{U}) \longrightarrow J^{k-1} V,$$

is also an immersion and the kernel of $T_Y \pi^k_{k-1}$ vanishes, i.e.

$$T_Y \pi^k_{k-1}(T_Y Q_Y) = 0,$$

where Q_Y is defined above.

Then

$$T_Y \pi^k_{k-1}(T_Y J\Phi \cap T_Y Q_Y) = T_Y \pi^k_{k-1}(C_Y(\Phi)) = 0,$$

in consequence, the symbol $C_Y(\Phi)$ of Φ at $Y \in J\Phi$ vanishes, and condition 1) of Theorem 1 is verified.

To verify that $\pi^{k+1}_k : J(P\Phi) \longrightarrow J\Phi$ is a local submersion we consider the commutative diagram:

$$\begin{array}{ccc} C^{1,k} V & \xrightarrow{i^{1,k}} & C^1(C^k V) \\ {\scriptstyle \Psi^{k+1}} \uparrow & & \uparrow {\scriptstyle \Psi^{1,k}} \\ J^{k+1} V & \xrightarrow[\tilde{i}^{1,k}]{} & J^1(J^k V) \end{array}$$

where,

$$\tilde{i}^{1,k} : j^{k+1}_u g \in J^{k+1} V \longmapsto j^1_{j^k_u g}(j^k g) \in J^1(J^k, U, \alpha)$$

is the natural immersion [4].

Now using Proposition 1 we have,

$$Z \in J(P\Phi) \Leftrightarrow \Psi^{1,k}(Z) \in C^{1,n}W \cap C^{k+1,n}V = PW.$$

Hence $\Psi^{1,k}(J\Phi) = PW$.

Now, by hypothesis:

$$\rho_k^{k+1} : PW \longrightarrow W$$

is a local submersion and

$$\Psi^k \circ \pi_k^{k+1} = \rho_k^{k+1} \circ \Psi^{k+1}.$$

Consequently:

$$\pi_k^{k+1} : P(J\Phi) \longrightarrow J\Phi,$$

is a local submersion, and condition 2) of Theorem 1 is satisfied.

Finally, Φ is complete in Y because it is a sheaf of ideals generated by functions which vanish on the regular manifold $(\Psi)^{-1}(W \cap \mathbf{V})$.

It follows that Φ is a completely integrable system of partial differential equations. Hence, there exists a solution γ of Φ such that:

$$\gamma(0) = x, \quad j_0^k\gamma = X, \quad \& \quad j_u^k\gamma \in J\Phi, \quad u \in U_\gamma.$$

Let $S = \gamma(U_\gamma)$, then clearly S verifies:

$$x \in S, \quad C_x^kS = X, \quad C^kS \subset W,$$

in consequence S is a solution of W at $X \in W$.

If $\tilde{S}$ is another solution of W at X, then $C_x^kS = C_x^k\tilde{S}$.

In particular $T_xS = T_x\tilde{S}$ and therefore there are a fibred manifold (V, U, ρ), associated to X, and parametrizations γ, $\tilde{\gamma}$ of S and $\tilde{S}$, respectively, which are sections of the fibred manifold (V, U, ρ).

Since S (resp. $\tilde{S}$) is solution of W at X we have: $j^k\gamma \subset J\Phi$ (resp. $j^k\tilde{\gamma} \subset J\Phi$), and $j_0^k\gamma = j_0^k\tilde{\gamma}$. Then by uniqueness of germ solutions of completely integrable systems, there exits an open set $B \subset \mathfrak{R}^n$ such that $\gamma|_B = \tilde{\gamma}|_B$.

Let $A = \rho^{-1}(B)$, then $S \cap A = \tilde{S} \cap A$. □

Remark. For $k = 1$ the submanifold W defines a Frobenius system. Indeed:

By condition 1) $\rho_0^1 : \longrightarrow M$ is a local immersion in a neighborhood $\mathbf{V}$ of X.

Then it is possible to define a section $\sigma = (\rho_0^1|_{W\cap\mathbf{V}})^{-1}$. Also we can find coordinate neighborhoods (V, x_i, j^j) and $(\mathbf{V}, x_i, y^j, p_i^j)$ of x and X respectively, such that the section σ can be expressed as,

$$\sigma(x_i, y^j) = (x_i, y^j, F_i^j(x_i, y^j)), \quad \text{with} \quad F_j^i : \mathbf{V} \longrightarrow \mathfrak{R}.$$

Then the manifold W defines a differentiable distribution D generated by:

$$L_i = \frac{\partial}{\partial x_i} + \sum F_i^j \frac{\partial}{\partial y^j} \ .$$

This distribution is involutive if and only if:

$$\frac{\partial F_i^j}{\partial x^k} + \sum_{l=1}^{m} \frac{\partial F_i^j}{\partial y^l} F_k^l = \frac{\partial F_k^j}{\partial x^i} + \sum_{l=1}^{m} \frac{\partial F_k^j}{\partial y^l} F_i^l \ .$$

By condition 2) we have:

$$\rho_1^2 : C^{1,n}W \cap C^{2,n}M \longrightarrow W.$$

is a local submersion.

Then given $Z \in W$, there exits $Z^2 \in C^{1,n}W \cap C^{2,n}M$ such that $\rho_1^2(Z^2) = Z$.

Since $Z \in C^2M$ we have:

$$p_{ik}^j(Z) = p_{ki}^j(Z).$$

Moreover, $Z \in C^{1,n}W$ then we have:

$$Z^2 = (x_i, y^j, F_i^j, \partial_{x_k}^{\#} F_i^j), \quad \partial_{x_k}^{\#} F_i^j = \frac{\partial F_i^j}{\partial x_k} + \sum_{l=1}^{m} \frac{\partial F_i^j}{\partial y^l} F_k^l \ .$$

In consequence,

$$p_{ik}^j(Z^2) = \partial_{x_k}^{\#} F_i^j(x_i, y^j) = \partial_{x_i}^{\#} F_k^j(x_i, y^j) = p_{ki}^j(Z^2),$$

and the involutivity of the distribution D defined above is verified.

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(Received May 26, 1994; revised February 28, 1995)