On the approximation of measurable functions.

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Recently in a paper on a conjecture of H. STEINHAUS one of us [3] proved the following

Theorem A. *Let* $\{f_n(x)\}$ $(n=1,2,\ldots)$ *be a sequence of measurable functions defined over the interval* $[0,1]$, *such that* $0\leqq f_n(x)\leqq 1$ *for any* $n=1,2,\ldots$. *Suppose further that this system is maximal, that is for* $x_1\neq x_2$ *there exists at least one function of the sequence, which takes different values at these points, provided that neither* x_1, *nor* x_2 *belong to a certain exceptional set* Z *of measure zero. Then the set of all functions of the form* $f_1^{\nu_1}(x)f_2^{\nu_2}(x)\ldots f_n^{\nu_n}(x)$ $(\nu_k\geqq 0$ *entire,* $n=1,2,\ldots)$ *is closed in the space* L^2 [1])

The notion of maximal systems of functions occurs in a theorem of M. H. STONE [4] which — using our terminology and for functions defined on the interval $[0,1]$ — can be stated as follows: if $\{f_n(x)\}$ $(n=1,2,\ldots)$ is a maximal system of continuous functions, then every continuous function $f(x)$ can be uniformely approximated by polynomials $P_m(f_1(x),f_2(x),\ldots,f_m(x))$ of the functions $f_k(x)$, with arbitrary accuracy, if m is sufficiently large.[2])

The purpose of the present paper is to give a new proof, and at the same time a generalization of Theorem A, namely to prove the following:

Theorem B. *Let* $\{f_n(x)\}$ $(n=1,2,\ldots)$ *be a sequence of measurable functions in the interval* $[0,1]$, *satisfying* $0\leqq f_n(x)\leqq 1$. *Let us suppose that the system is maximal in the sense given in Theorem A. Then for any measurable function* $f(x)$, *for which almost everywhere in* $[0,1]$ $a\leqq f(x)\leqq b$, *for any* $\varepsilon>0$ *and* $\delta>0$ *and for a sufficiently large* m *we can determine a polynomial* $P_m(f_1(x),f_2(x),\ldots f_m(x))$ *of the functions* $f_1(x),f_2(x),\ldots f_m(x),\ldots$ *such that*

$$(1)\qquad |f(x)-P_m(f_1(x),f_2(x),\ldots,f_m(x))|<\varepsilon$$

1) Theorem A holds also if instead of the boundedness of the functions $f_n(x)$, it is supposed only that they all belong to every space $L^p, p>1$.

2) In the theorem of STONE instead of the interval $[0,1]$ any compact topological space may be taken.

for x *belonging to a set* E *of measure greater then* $1-\delta$, *further we have*

$$a \leqq P_m(f_1(x), f_2(x), \ldots, f_m(x)) \leqq b \tag{2}$$

for every value of x *in* $[0, 1]$.

Theorem A follows from Theorem B evidently as if $a \leqq f(x) \leqq b$, and (1) and (2) hold, we have, denoting by $\bar{E}$ the complementary set of E,

$$\int_0^1 (f-P_m)^2\,dx \leqq \varepsilon^2 + \int_{\bar{E}} (f-P_m)^2\,dx \leqq \varepsilon^2 + (b-a)^2\delta$$

and if $\delta < \frac{\varepsilon^2}{(a-b)^2}$ we have

$$\int_0^1 (f-P_m)^2\,dx < 2\varepsilon^2.$$

But as the set of bounded functions is everywhere dense in [0, 1], Theorem A follows. The idea of our proof of Theorem B is applicable to prove the Theorem of STONE also; and it seems to us that this method of proof may be of some interest in itself also. Let us mention that Theorem B, without the conclusion (2), follows from the theorem of STONE, by a simple application of a theorem of N. LUSIN [1]. As a matter of fact, we can find, by using LUSIN's theorem, a closed subset G of $[0, 1]$ and a set of continuous functions $\{g_n(x)\}$ such that the measure of G is $> 1-\delta$ and $g_n(x) = f_n(x)$ for $x \varepsilon G$ and $n = 1, 2, \ldots$. Applying the theorem of STONE to the set G, and the system $\{g_n(x)\}$ which is clearly maximal on G, we obtain assertion (1) of Theorem B, but not (2), because by applying the theorem of STONE to the set G, no conclusion regarding the behaviour of the polynomials P_m outside G, can be made.

Proof of Theorem B. In the proof instead of the maximality of the system $\{f_n(x)\}$ we shall use only the following somewhat weaker condition: for any $\eta > 0$ a set E_1 of measure $> 1-\eta$ can be found on which the system is maximal.

Let E_1 be such a set for $\eta = \frac{\delta}{4}$. [3])

In what follows, by $\mu(E(\ldots))$ we shall denote the measure of the set $E(\ldots)$ of points satisfying the condition $(\ldots)$. Applying the theorem of N. LUSIN [1], we can construct a sequence of continuous functions $\{g_n(x)\}$ defined over [0, 1] such that $\mu(E(f_n(x) \neq g_n(x))) < \frac{\delta}{2^{n+1}}$ and $0 \leqq g_n(x) \leqq 1$, $(n = 1, 2, \ldots)$.

Thus we substitute our original system by a system of uniformly bounded continuous functions, and the corresponding functions of the two

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[3]) In other words the set E_1 is chosen so that it contains no exceptional points.

system coincide on a set E_2 of a measure greater then $1-\frac{\delta}{2}$. Now we can choose applying again the theorem of LUSIN a subset E_3 of E_2 with $\mu(E_3)>1-\frac{3\,\delta}{4}$ such that $f(x)=g(x)$ on E_3, where $g(x)$ is a continuous function, and $a\leqq g(x)\leqq b$ for $x\in[0,1]$.

Now we can find a closed subset E of E_3 with $\mu(E)>1-\delta$. Let us consider the compact space $H^\infty=\prod_{r=1}^{\infty} I_r$ defined as the infinite topological product of the intervals $I_r=[0\leqq x_r\leqq 1]$. (For the topological notions used see [2].) The correspondence $x\rightarrow(g_1(x),g_2(x),\dots,g_n(x),\dots)$ defines a continuous one-to-one mapping of the set E into a subset L of the space H^∞. Since E is compact, this is a homeomorphism between E and L, and therefore L is closed. Now let us define a continuous function $h(P)$ on L, as follows: if $P\in L$ corresponds to a point $x\in E$, let us put $h(P)=g(x)$. The function $h(P)$ is clearly continuous on L, and $a\leqq h(P)\leqq b$. Applying TIETZE's theorem (see [2] p. 73, the condition of normality for H^∞ being manifestly satisfied), we can extend this function to a continuous function defined over the space H^∞, with values between a and b. Let us denote this further by $F(P)$ $(P\in H^\infty)$. The following easy reasoning shows that for every $\varepsilon>0$ and for a sufficiently large m, there exists a continuous function $F_m(Q)$ in the subspace $H^m=\prod_{r=1}^{m} I_r$ such that

$$|F(P)-F_m(Pr_{H^m}P)|<\frac{\varepsilon}{2} \tag{3}$$

(here $Pr_{H^m}P$ denotes the projection of P into H^m) and $a\leqq F_m(Q)\leqq b$. This can be seen as follows.

By the compactness of H^∞ and by the continuity of $F(P)$ for every $\varepsilon>0$ we can find a finite system of neighborhoods V_r $(r=1,2,\dots,N)$ covering H^∞ such that if $P_1\in V_r$ and $P_2\in V_r$ then $|F(P_1)-F(P_2)|<\frac{\varepsilon}{2}$. We can make this so that each neighborhood has the form $\prod_{r=1}^{\infty} O_r$, where the open interval O_r coincides with I_r if r is sufficiently large. Hence we can find a number m_0 such that for every pair of points P_1 and P_2 in H^∞, from $P_{r_{H^m}}P_1=P_{r_{H^m}}P_2$, where $m\geqq m_0$ we may conclude that P_1 and P_2 fall in the same neighborhood of the system $\{V_r\}$ $(r=1,2,\dots N)$.

We denote by Y^m the product $\prod_{r>m} I_r$ and define the function $F_m(Q)$ $(Q\in H^m)$ as follows: we choose an arbitrary but fixed point R in Y^m, and determine for any $Q\in H^m$ the unique point P of H^∞ for which $P_{r_{H^m}}P=R$

and $Pr_{Y^m} P = R$, finally we put $F_m(Q) = F(P)$. The function $F_m(Q)$ obviously has all the required properties.

By the WEIERSTRASS approximation theorem, for every $\varepsilon > 0$ we can find a polynomial $P_m(x_1, x_2, \ldots, x_m)$ in H^m with $a \leqq P_m \leqq b$, and such that if $Q = (x_1, x_2, \ldots x_m) \in H^m$

(4) $$|F_m(Q) - P_m(x_1, x_2, \ldots, x_m)| < \frac{\varepsilon}{2},$$

and $a \leqq P_m(x_1, x_2, \ldots x_m) \leqq b$. Hence by (3) and (4), if $Pr_{H^m} P = Q = (x_1, x_2, \ldots, x_m)$, we have

(5) $$|F(P) - P_m(x_1, x_2, \ldots, x_m)| < \varepsilon.$$

Let us substitute $g_n(x)$ for x_n i. e. choose $P \in L$. If $x \in E$ then $g_n(x) = f_n(x)$ and $g(x) = f(x)$ and therefore $P_m(x_1, x_2, \ldots, x_m) = P_m(f_1(x), f_2(x), \ldots, f_m(x))$ and $F(P) = f(x)$ and thus it follows from (5) that

(6) $$|f(x) - P_m(f_1(x), f_2(x), \ldots, f_m(x))| < \varepsilon \text{ for } x \in E$$

and we have $\mu(E) > 1 - \delta$, further $a \leqq P_m(f_1(x), f_2(x), \ldots, f_m(x)) \leqq b$. Thus Theorem B is proved.

It is easy to see that the above method of proof furnishes also a new and natural approach to the theorem of M. H. STONE mentioned above, for the case of the interval (0, 1). [4]) In the case of an arbitrary compact space the proof works also, the only essential novelty is, that for proving the compactness of the space H^∞ we must refer to a theorem of Tychonoff [5].

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[4]) As *B. Szőkefalvi Nagy* kindly informed us, a part of our reasoning occurs also in other connection in § 4 of his paper: Zur Theorie der Charactere Abelscher Gruppen, *Math. Ann.* **114** (1937), 373—384.