Simply harmonic affine spaces of symmetric connection.

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§ 1. Harmonic Riemannian spaces.

Consider, in the small, a Riemannian n-space R^n for which

$$ds^2 = e g_{ij} dx^i dx^j \qquad (e = \pm 1), \tag{1.1}$$

the metric being not necessarily positive definite. Let P_0, (x_0^i), be a fixed point of R^n, and P, (x^i), a variable point, and let $s \equiv s(x_0, x)$ be the length of the arc of the geodesic, assumed to exist and to be unique, joining P_0, P. Further, let [1])

$$\Omega = \frac{1}{2} e s^2, \tag{1.2}$$

where e is the indicator of the geodesic P_0P. Then R^n is called *centroharmonic*[2]) with respect to the base-point P_0 if $\Delta_2 \Omega \equiv \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(g^{ij} \sqrt{|g|} \frac{\partial \Omega}{\partial x^j}\right)$ is a function of Ω only, say

$$\Delta_2 \Omega = f(\Omega),$$

not involving x_0^i or x^i explicitly. The R^n is called *completely harmonic* if this holds for all base-points P_0. If the function f happens to be a constant, the space is called *simply centroharmonic* or *simply harmonic* according as the relation holds for one base-point or for all base-points. The constant value of f is then necessarily equal to n.

Very little is yet known about harmonic spaces in the large. Many of them, indeed, are of indefinite metric, and global theories of Riemannian spaces of indefinite metric are almost non-existent. Recent work on fibre bundles has broken new ground, but the present paper is confined to local properties of the spaces discussed.

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[1]) I have been accustomed in previous papers to omit the indicator e in (1. 1) and (1. 2), allowing s to be imaginary along geodesics for which $e = -1$: cf. SYNGE, *Proc. London Math. Soc.*, **32** (1931), 242. I now follow A. G. WALKER in inserting the indicator e, with the understanding that s is always real. The same function Ω is thus defined. When the geodesic P_0P is null, the function $s(x_0, x)$ still exists as an analytic entity, the equation $s(x_0, x) = 0$ being that of the null cone of P_0.

[2]) In previous papers the term used has been *centrally harmonic*.

The concept of a completely harmonic Riemannian space, depending as it does upon the idea of geodesic arc-length, does not extend naturally to non-metrical spaces. It has recently been noticed by E. M. PATTERSON,[3]) however, that the infinite sequence of relations[4]) satisfied in a *simply* harmonic analytic Riemannian space by the curvature tensor and its derivatives can be used, *mutatis mutandis,* to define a simply harmonic affine space of symmetric connection. It is the purpose of this paper to give a direct definition of such a space, whether analytic or not, and to apply it to re-establish a theorem of PATTERSON on the Riemann extension of an affine space.

§ 2. Definition of a simply harmonic affine space of symmetric connection.

Let A^n be an n-dimensional affine space of coordinates (x^i) and symmetric connection Γ^i_{jk}. The differential equations of the paths, referred to an affine parameter t, are

$$(2.1) \qquad \frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

Consider the path through P_0, (x^i_0), of given direction $a^i = \left(\dfrac{dx^i}{dt}\right)_0$ at P_0. Take $t = 0$ at P_0. The equations of the path are of the form

$$(2.2) \qquad x^i = \Phi^i(x_0, at),$$

where x_0, a are written for $(x^1_0, \ldots, x^n_0)$ and $(a^1, \ldots, a^n)$. If we put

$$(2.3) \qquad y^i = a^i t,$$

then (2.2) becomes

$$(2.4) \qquad x^i = \Phi^i(x_0, y)$$

and gives the transformation[5]) from the x-coordinates to Riemannian normal coordinates y^i of origin P_0, (2.3) being the equation in normal coordinates of the path a^i.

Let $Y^i(x_0, x)$ be the contravariant vector defined by

$$(2.5) \qquad Y^i = t\frac{dx^i}{dt},$$

where the right-hand side is calculated from (2.2) and the $a^i t$ are then eliminated by means of (2.2) to give Y^i as a function of the x^i_0 and x^i. Then:

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[3]) E. M. PATTERSON, *J. London Math. Soc.,* **27** (1952), 102—107.

[4]) See, e. g., A. G. WALKER, *Proc. Edinburgh Math. Soc.,* **7** (1942), 25.

[5]) If the allowable transformations of coordinates in A^n are of class C^r, r finite, then the coefficients of connection are to be assumed of class C^{r-2}, and the transformation (2.4) to normal coordinates also of class C^{r-2}. Normal coordinates for a space of finite class may be called *specially* allowable, a term used in a similar connection by MARSTON MORSE, Calculus of variations in the large, (1934), p. 108.

Definition. The affine space A^n of symmetric connection Γ^i_{jk} is *simply centroharmonic* with respect to the base-point P_0 if

$$\Upsilon^i_{;i} = n, \tag{2.6}$$

the semicolon denoting covariant differentiation with respect to the x's and the connection Γ^i_{jk}, and is *simply harmonic* if

$$\Upsilon^i_{;i} \equiv n \tag{2.7}$$

for all base-points P_0.

When the A^n is Riemannian, the Γ^i_{jk} being Christoffel symbols and t the arc-length s, this definition agrees with that of a simply harmonic Riemannian space. For in that case, as is well known,

$$\frac{\partial s}{\partial x^i} = e g_{ij} \frac{dx^j}{ds} \tag{2.8}$$

for any given geodesic, whence, multiplying by es,

$$\Omega_{,i} = g_{ij}\, s \frac{dx^j}{ds},$$

the comma denoting covariant differentiation with respect to the Christoffel symbols. Raising the suffix i, we get

$$\Omega^i = s \frac{dx^i}{ds},$$

which may be compared with (2. 5). Since $\Delta_2 \Omega = \Omega^i_{,i}$, the agreement of (2. 7) with the definition for Riemannian spaces is apparent. That Υ^i plays a part in an affine space similar to that played by Ω^i in a Riemannian space was noted by SYNGE.[1])

A simply harmonic A^n will be called an SA^n.

Using an asterisk to denote the components of any geometric object in the normal coordinate system y^i, we have, by (2. 5) and (2. 3),

$$\begin{aligned} {}^*\Upsilon^i &= t \frac{dy^i}{dt} \\ &= y^i. \end{aligned} \tag{2.10}$$

Hence

$$\begin{aligned} {}^*\Upsilon^i_{;i} &\equiv \frac{\partial\, {}^*\Upsilon^i}{\partial y^i} + {}^*\Gamma^i_{ji}\, {}^*\Upsilon^j \\ &= n + {}^*\Gamma^i_{ji}\, y^j, \end{aligned} \tag{2.11}$$

so the condition for A^n to be simply centroharmonic with respect to P_0 is

$${}^*\Gamma^i_{ji}\, y^j = 0. \tag{2.12}$$

If the A^n is analytic, ${}^*\Gamma^i_{ji}\, y^j$ may be expanded in a well known way[6]) as a

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[6]) O. VEBLEN, Invariants of quadratic differential forms, *Cambridge Math. Tract*, 24 (1924), p. 90, (6. 4).

series in the y^i, the coefficients being affine normal tensors evaluated at P_0. Because of (2. 12), the coefficients in this series are all zero, and we obtain the infinite sequence of relations [7]) used by PATTERSON, though in a different form, as the basis of the definition of an SA^n.

Y^i is a contravariant vector at the point (x^i) whose components in the normal coordinate system happen to be y^i. It is convenient here to introduce the n functions $Y^{(i)}(x_0, x)$, distinguished from the components of the vector $Y^i(x_0, x)$ by having a bracketed suffix, defined as follows. Let the transformation inverse to (2. 4) be

$$y^i = \psi^i(x_0, x),$$

and define

$$Y^{(i)} = -\psi^i(x_0, x).$$

Thus

$$Y^{(i)} = -y^i \qquad (2.13)$$
$$= -a^i t,$$

that is,

$$Y^{(i)} = -t\left(\frac{dx^i}{dt}\right)_0. \qquad (2.14)$$

Thus the $Y^{(i)}$ are the components of a vector *at* P_0, transforming, under $x \to \bar{x}$, according to

$$\bar{Y}^{(i)} = Y^{(j)}\left(\frac{\partial \bar{x}^i}{\partial x^j}\right)_0,$$

unlike Y^i, whose law of transformation involves the derivatives $\partial \bar{x}^i / \partial x^j$ evaluated *at* P. In other words, $Y^i(x_0, x)$ behaves like a scalar with respect to x_0^i and a vector with respect to x^i, and vice-versa for $Y^{(i)}(x_0, x)$. It is easy to see that, in fact, $Y^{(i)}$ is obtainable from Y^i by an interchange of x_0^i and x^i, that is,

$$Y^{(i)}(x_0, x) = Y^i(x, x_0). \qquad (2.15)$$

§ 3. Riemann extension of an affine space of symmetric connection.

As before, let A^n be an affine space of coordinates (x^i) and symmetric connection Γ^i_{jk}. Let c_{ij} be a given symmetric covariant tensor in A^n, and let R^{2n} be the Riemannian space of $2n$ dimensions, of [8]) coordinates $(x^\alpha) \equiv (x^i, x^{i'})$, for which

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \equiv (c_{ij} - 2\Gamma^h_{ij}\, \xi_h)\, dx^i\, dx^j + 2dx^i\, d\xi_i, \qquad (3.1)$$

where ξ_i has been written for $x^{i'}$, the term $2dx^i\, d\xi_i$ meaning $2\Sigma\, dx^i\, dx^{i'}$. The

[7]) Cf. E. T. COPSON and H. S. RUSE, *Proc. Roy. Soc. Edinburgh*, **60** (1939—40), 130, or A. LICHNEROWICZ, *Bull. Soc. Math. de France*, **72** (1944), 156.

[8]) Unprimed Latin suffixes will continue to run from 1 to n. Primed Latin suffixes will run from $n+1$ to $2n$, with the understanding that $i' = n+i$. Greek suffixes will run from 1 to $2n$.

space R^{2n} is called [9]) a *Riemann extension* of the affine space A^n. Under a transformation of coordinates $x \rightarrow \bar{x}$ in A^n, the form of ds^2 is preserved provided that ξ_i is transformed as a covariant vector of A^n, c_{ij} as a tensor and Γ^h_{ij} as an affine connection.

PATTERSON [3]) has shown that, if R^{2n} is simply harmonic, so is A^n, and conversely. In this section I obtain a formula for Ω in R^{2n}, and use it to prove a slightly stronger form of Patterson's theorem, namely that, *if R^{2n} is harmonic, then it is simply harmonic and so is A^n*; with PATTERSON's converse that, *if A^n is simply harmonic, so is R^{2n}*.

The fundamental tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$ in R^{2n} are given by

$$g_{ij} = c_{ij} - 2\Gamma^h_{ij}\xi_h,\; g_{ij'} = g_{j'i} = \delta_{ij},\; g_{i'j'} = 0; \tag{3.2}$$

$$g^{ij} = 0, \qquad g^{ij'} = g^{j'i} = \delta^{ij},\; g^{i'j'} = -g_{ij}; \tag{3.3}$$

while

$$g = \det|g_{\alpha\beta}| = (-1)^n, \tag{3.4}$$

the δ's being Kronecker symbols.

Of the Christoffel symbols $\left\{ {\alpha \atop \beta\gamma} \right\}$ of R^{2n}, those for which $\alpha, \beta, \gamma = 1, 2, \ldots, n$ are given by

$$\left\{ {i \atop jk} \right\} = \Gamma^i_{jk}.$$

Hence the first n of the differential equations

$$\frac{d^2x^\alpha}{ds^2} + \left\{ {\alpha \atop \beta\gamma} \right\} \frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \tag{3.5}$$

of the geodesics of R^{2n} are the differential equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \tag{3.6}$$

of the paths of A^n, with s as affine parameter.

Let $Q_0, (x^\alpha_0) \equiv (x^i_0, x^{i'}_0) \equiv (x^i_0, \xi^0_i)$, be a fixed point of R^{2n}, and consider the geodesic joining this to the point $Q, (x^\alpha) \equiv (x^i, \xi_i)$. A first integral of equations (3.5) is $g_{\alpha\beta}\dfrac{dx^\alpha}{ds}\dfrac{dx^\beta}{ds} = e$, that is,

$$e = (c_{ij} - 2\Gamma^h_{ij}\xi_h)\frac{dx^i}{ds}\frac{dx^j}{ds} + 2\frac{dx^i}{ds}\frac{d\xi_i}{ds}.$$

By (3.6) this is

$$e = c_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} + 2\frac{d}{ds}\left\{\xi_i\frac{dx^i}{ds}\right\}.$$

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[9]) E. M. PATTERSON and A. G. WALKER, *Quarterly J. of Math.* (Oxford) (2), **3** (1952), 19—28.

Integrating with respect to s from Q_0 to Q, we obtain a formula for the length s of the geodesic Q_0Q, namely

$$es = \frac{2}{s} F(x_0, x) + 2 \left\{ \xi_i \frac{dx^i}{ds} - \xi_i^0 \left(\frac{dx^i}{ds} \right)_0 \right\}, \tag{3. 7}$$

where $F(x_0, x) \equiv \frac{s}{2} \int_0^s c_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} ds$ is a scalar depending, as is easy to see, upon x_0^i and x^i only, and not upon ξ_i or ξ_i^0.

Multiplying (3. 7) by $\frac{1}{2} s$, using (1. 1) and (2. 5), (2. 14) with t duly identified with s, we obtain, for R^{2n},

$$\Omega = F(x_0, x) + \xi_i \Upsilon^i + \xi_i^0 \Upsilon^{(i)}, \tag{3. 8}$$

Υ^i and $\Upsilon^{(i)}$, like F, being independent of ξ_i, ξ_i^0.

In R^{2n},

$$\begin{aligned} \Delta_2 \Omega &\equiv \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{|g|}\, g^{\alpha\beta} \frac{\partial \Omega}{\partial x^\beta} \right) \\ &= \frac{\partial}{\partial x^\alpha} \left(g^{\alpha\beta} \frac{\partial \Omega}{\partial x^\beta} \right) \quad \text{by (3, 4)}, \\ &= 2 \frac{\partial^2 \Omega}{\partial x^i \partial \xi_i} - \frac{\partial}{\partial \xi_i} \left(g_{ij} \frac{\partial \Omega}{\partial \xi_j} \right) \end{aligned}$$

by (3. 3) and the fact that $x^{i'} = \xi_i$. By (3. 8) this gives

$$\begin{aligned} \Delta_2 \Omega &= 2 \frac{\partial \Upsilon^i}{\partial x^i} - \frac{\partial}{\partial \xi_i} (g_{ij} \Upsilon^j) \\ &= 2 \frac{\partial \Upsilon^i}{\partial x^i} + 2 \Gamma_{ij}^i \Upsilon^j \quad \text{by (3. 2)}, \end{aligned}$$

that is,

$$\Delta_2 \Omega = 2 \Upsilon^i_{;i}. \tag{3. 9}$$

If R^{2n} is completely harmonic, $\Delta_2 \Omega$ is a function of Ω. But Ω depends upon ξ_i and ξ_i^0, while the right-hand side of (3. 9) does not. Hence $\Delta_2 \Omega$ must be a constant, and R^{2n} *simply* harmonic, and so

$$\Delta_2 \Omega = 2n, \tag{3. 10}$$

whence

$$\Upsilon^i_{;i} = n \tag{3. 11}$$

and A^n is simply harmonic. Conversely, if (3. 11) is true, so is (3. 10), and this completes the proof of Patterson's theorem and its converse. If R^{2n} is initially assumed to be merely centroharmonic with respect to (x_0^i) instead of completely harmonic, the theorem remains true if the word *harmonic* is prefixed everywhere by *centro*.

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