Remarks on isotopies.

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§ 1. A generalized commutative law.

Let $G(*) = G$ be a groupoid. $G(*)$ will be called *generalized Abelian* (abbreviated g. A.) provided there are four permutations P, Q, R, and S of G so that:

(1) $$xP^*yQ = yR^*xS; \qquad \text{(for all } x, y \text{ in } G).$$

The "law" [1] is suggested by the generalized associative law (restricted to permutations) of T. EVANS [6]. EVANS' law is:

(2) $$((xA^*yB)C^*zD)E = (xF^*(yG^*zH)I)J; \quad \text{(for all } x, y, z \text{ in } G),$$

where $A, B, C, D, E, F, G, H, I$ and J are permutations of G. The law [2] arises naturally if one seeks an isotopy invariant generalization of associativity. Analogously we have

Theorem 1. *If $G(*)$ and $H(+)$ are isotopic groupoids then $G(*)$ is g. A. if and only if $H(+)$ is g. A..*

Proof. Since the g. A. property is obviously an isomorphism invariant, it suffices by the principal isotopy theorem [2] to show that the proposition is valid for principal isotopes $G(*)$ and $G(+)$. Let (1) hold and let M and N be permutations of $G(*)$ with $x^*y = xM + yN$ for all x, y in $G(*)$. Then $xPM + yQN = yRM + xSN$ and $G(+)$ is g. A.. Since principal isotopy is an equivalence relation, the g. A. property for $G(+)$ implies the g. A. property for $G(*)$.

Corollary. *A necessary condition that a groupoid be isotopic to an Abelian groupoid is that it be g. A..*

Theorem 2. *If $G(*)$ has a unit e and is g. A., then there exists a permutation V of $G(*)$ such that*

(3) $$x^*y = yV^{-1*}xV \qquad \text{(for all } x, y \text{ in } G(*)).$$

Moreover, if eV is idempotent, $G(*)$ is Abelian.

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Proof. Let $G(*)$ have unit e and satisfy (1). Unless otherwise specified, a and x are arbitrary elements of $G(*)$. W is the identity mapping of $G(*)$. L_t and R_t denote, respectively, the left and right translations (in $G(*)$) of $G(*)$ by the element t. Products of mappings are to be read from left to right. From (1) we have

$$(4) \qquad x^* y = y U^* x V \quad \text{with} \quad U = Q^{-1} R \quad \text{and} \quad V = P^{-1} S.$$

Then, $L_a = U R_{aV}$ and $R_a = V L_{aU} \cdot V L_{eU} = R_e = W$ and $e V^{-1} = e L_{eU} = (eU)e = eU$. Applying (4) twice, one finds $R_a = VUR_{aUV}$ so that $R_e = VUR_{eUV} = VUR_e = VU = W$ and $U = V^{-1}$. Suppose now that $\mathring{e} V^* e V = e V$. $R_{eV} = VL_{eVU} = VL_e = V$. Thus, $eVR_{eV} = eV^* eV = eV$ and $eVV = eV$ so that $e = eV$ and $V = R_{eV} = R_e = W$.

Theorem 3. *A semigroup with unit which is g. A. is Abelian.*

Proof. Using (4) and the associativity one obtains $L_a = L_{eV^{-1}} R_{aR_{eV}} = R_a R_{eV} L_{eV^{-1}} = R_a W = R_a$.

Lemma 1. (T. Evans (6).) *If $G(*)$ is finite or is a quasigroup and if $G(*)$ has a unit, then $G(*)$ is associative if and only if $G(*)$ has a law* (2).

Combining lemma 1 with theorems 1 and 3 and theorem 1 A of Bruck's paper [2], and recalling that both the Evans law and the g. A. law are isotopy invariants, we obtain the following characterization theorem:

Theorem 4. *A quasigroup (finite groupoid with left and right non-singular elements) is isotopic to an Abelian group (Abelian semigroup) if and only if it is g. A. and is associative in the sense of* (2).

§ 2. Isotopy of semilattices.

Let $G(*)$ and $H(+)$ be semilattices [5] in what follows.

Theorem 5. *Any homotopy*[1]) *of $G(*)$ onto $H(+)$ induces a homomorphism of $G(*)$ onto $H(+)$ which is actually the single mapping of the homotopy.*

Proof. Let A, B, and C be (single-valued) mappings of $G(*)$ onto $H(+)$ so that $(a^*b)C = aA + bB$ for all a, b in $G(*)$. Now, $aC = (a^*a)C = aA + aB$. Thus, $aC + bC = (aA + aB) + (bA + bB) = (a^*b)C + (b^*a)C = (a^*b)C + (a^*b)C = (a^*b)C$ and C is a homomorphism.

Remark. The above proof uses all the assumptions concerning $G(*)$ and $H(+)$ except the associativity of $G(*)$.

Corollary. *Two semilattices are isotopic if and only if they are isomorphic.*

Remark. The equivalence of isotopy and isomorphism for semigroups with unit is well known [2]. As might be anticipated, idempotency in both

[1]) See [3].

and commutativity in one of two isotopic semigroups ensure isomorphism regardless of the existence of units.

It has been observed several times in the literature (cf. [1], [4], [5]) that two lattices are isomorphic if and only if their join (meet) semilattices are isomorphic. The author has shown that a semilattice admits a second operation to form a lattice if and only if it possesses the property M defined in [5]. M is obviously an isomorphism invariant. Thus we have

Theorem 7. *A semilattice admits a second operation to form a lattice if and only if it is isotopic to a semilattice of a lattice and in this case the two lattices are isomorphic.*

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