Contribution to lattice theory.

By K. ISEKI in Osaka (Japan).

Introduction.

Several authors have considered the characterization of special classes of lattices: modular, distributive, Boolean, etc. Recently the theory has been developed by Indian mathematicians and K. MATSUMOTO, T. MICHIURA. This paper is also devoted to a discussion of the same problem.

The work is divided into four sections. In the first section, we shall give a brief summary of well known notions and results which are needed in the sequel. Some of them are found in the articles of G. BIRKHOFF [1][1]) G. BIRKHOFF—O. FRINK [1], N. BOURBAKI [1] and K. ISEKI [5]. In the second section, we shall show that a distributive lattice may be characterized by a meet-irreducible filter. The result was published in my note [1] with a brief proof. Further, we shall discuss certain results of M. F. SMILEY—E. PITCHER [1] and G. PICKERT [1]. In the third section, we shall give the condition for a distributive lattice with 0 and 1 to be a Boolean algebra, by use of the results of the second section. In the final section O. FRINK's result [1] is generalized to atomic lattices.

In this article the terminology and notation introduced by G. BIRKHOFF [1] will be used without any further reference.

§ 1. Preliminary notions.

In this section, we shall consider the elementary facts relating to our discussion.

Let L be a lattice with least element 0. A filter F is a subset of the lattice L which satisfies the following conditions:

1. $0 \notin F$.
2. If $a \in F$ and $x \geqq a$, then $x \in F$.
3. If $a \in F$, $b \in F$, then $a \cap b \in F$.

(For the concept of a filter see N. BOURBAKI [1] or P. SAMUEL [1]).

[1]) Numbers in brackets refer to the bibliography at the end of this paper.

Definition 1. *A filter of a lattice L is said to be prime if* $a \cup b \in F$ *implies* $a \in F$ *or* $b \in F$.

Definition 2. *A filter U of a lattice L is said to be maximal or ultrafilter if there exists no filter containing U.*

Definition 3. *A filter F is said to be meet irreducible if it is not the set intersection of two filters each* $\neq F$.

The following lemma is due to G. BIRKHOFF and O. FRINK [1].

Lemma 1. *Any prime filter is meet-irreducible. Conversely, in a distributive lattice, all the meet-irreducible filters are prime.*

Proof. Let F be a prime filter. If it is not meet-irreducible, there exist two filters A, B such that

$$F = A \cap B, \quad A \neq F \neq B.$$

Therefore there exist two elements a, b such that

$$a \notin F, \quad a \in A \text{ and } b \notin F, \quad b \in B.$$

Since A, B are filters, $a \cup b \in A \cap B = F$. By definition 1, $a \in F$ or $b \in F$. This shows that F is meet-irreducible.

Conversely, let F be meet-irreducible but not prime in a distributive lattice L. We can take two elements a, b such that

$$a \cup b \in F, \quad a \notin F, \quad b \notin F.$$

Define two filters $F * a$, $F * b$ by

$$F * a = \{x | x \geqq a \cap f; f \in F\}, \quad F * b = \{x | x \geqq b \cap f; f \in F\}.$$

We have $(F * a) \cap (F * b) \supset F$. Let x be any element of the set $(F * a) \cap (F * b)$, then we have $x \geqq a \cap f, f' \cap b, f, f' \in F$ and

$$x \geqq (f \cap f' \cap a) \cup (f \cap f' \cap b) = (f \cap f') \cup (a \cap b).$$

This means $(F * a) \cap (F * b) = F$. The lemma is therefore proved.

Definition 4. *We say that a lattice L has* WALLMAN *property (briefly W-property) if for* $a > b$, *there exists an* x *such that* $a \cap x \neq 0$, $b \cap x = 0$.

Definition 5. *L is called U-separated, if there exists for any two of its elements an ultrafilter containing one element, but not the other.*

Similarly we can define p-separated elements as follows.

Definition 6. *L is called p-separated if for any two of its elements there exists a prime filter containing one element, but not the other.*

These notions were first introduced in my recent articles [7], [8]. By this "U-separatedness", we can prove the

Theorem 1. *Every pair of distinct elements of a lattice L is U-separated if and only if L has the W-property.*

Proof. For details, see K. ISEKI [5].

§ 2. Characteristic properties of distributive lattices.

In this section, we shall prove the theorem mentioned in the introduction. For this purpose. we shall prove the following

Lemma 2. *If a principal filter $F_a = \{x | a \leqq x\}$ does not contain an element b, there exists a meet-irreducible filter G such that $F_a \subset G$ and $b \notin G$.*

Proof. The class of all filters containing F_a is ordered by set inclusion. Consider the linear ordered set $F_\alpha (\alpha < \Omega)$ in this class, and the set sum $\bigcup_\alpha F_\alpha$, then $\bigcup_\alpha F_\alpha$ is a filter which contains F_a but not b. The class, ordered by set inclusion, is inductive. By ZORN's lemma, there exists a filter G with the maximal property: $F_a \subset G$ and $b \notin G$. The lemma will be proved if we show that G is meet-irreducible. Let G be the set intersection of two filters G_1, G_2. If $b \in G_1, G_2$, then $G = G_1 \cap G_2$. Consequently one of them does not contain b. Suppose $b \notin G_1$; then $G = G_1$. This shows that G is meet-irreducible.

Theorem 2. *A necessary and sufficient condition for a lattice with 0 to be distributive is that every meet-irreducible filter be prime.*

Proof. It is obvious from Lemma 1 that condition is necessary. To prove the converse, we shall verify the ORE condition [2]: $a \cup x = b \cup x$, $a \cap x = b \cap x$ imply $a = b$. Suppose $a \neq b$, then either the principal filter F_a of a does not contain b or this is the case for the principal filter F_b and a. If $b \notin F_a$, by Lemma 2 there exists a meet-irreducible filter G containing F_a but not b. Since by our hypothesis G is prime, $a \cup x = b \cup x \in G$, $b \notin G$ imply $x \in G$. This shows that $a \cap x = b \cap x \in G$. Therefore $b \in G$, which is a contradiction. Similarly, $a \notin F_b$ leads to a contradiction. Hence $a = b$. This completes the proof.

M. F. SMILEY and E. PITCHER [1] generalized the GLIVENKO [1] definition of metric betweenness for an arbitrary lattice L. For three elements $a, b, c \in L$, b is between a and c if and only if

$$(a \cap b) \cup (b \cap c) = b = (a \cup b) \cap (b \cup c).$$

We shall use the notation abc to show that b is between a and c. One of their results was to characterize distributive lattices by a relation between DUTHIE's segment and betweenness. Following W. D. DUTHIE, we define the segment $\langle a, b \rangle$ of a, b as the set $\{x | a \cap b \leqq x \leqq a \cup b\}$.

Theorem 3. *A lattice L is distributive if and only if for every pair $a, b \in L$, $\langle a, b \rangle \ni x$* implies axb. (See M. F. SMILEY—E. PITCHER [1].)

Proof. Let L be a distributive lattice, then $a \cap b \leqq x \leqq a \cup b$ implies $(a \cap x) \cup (b \cap x) = x \cap (a \cup b) = x$ and also dually $(a \cup x) \cap (b \cup x) = x$.

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[2]) See O. ORE [1] or V. GLIVENKO [1].

Conversely, consider $a, b, x \in L$ such that $a \cup x = b \cup x$, $a \cap x = b \cap x$, then we have

$$a \cap x = b \cap x \leqq b \leqq b \cup x = a \cup x,$$
$$b \cap x \leqq a \leqq b \cup x.$$

Thus $b \in \langle a, x \rangle$, $a \in \langle b, x \rangle$. By the hypothesis we have

$$b = (a \cap b) \cup (b \cap x) = (a \cap b) \cup (a \cap x) = a.$$

Therefore L is distributive.

Theorem 4. *A lattice is distributive if and only if from*

(1) $$a, b \leqq c \cup d.$$

(2) $$a \cap c = b \cap c$$

(3) $$(a \vee c) \wedge d = (b \cup c) \wedge d,$$

there follows $a = b$.

A similar theorem on modular lattices has recently been proved by G. PICKERT [1].

Proof. Let L be a distributive lattice satisfying (1) (2) and (3). Then

$$c \vee ((a \vee c) \wedge d) = (c \vee (a \vee c)) \cap (c \cup d) = (a \cup c) \cap (c \cup d) = a \cup c$$
$$c \vee ((b \vee c) \wedge d) = (c \cup (b \cup c)) \cap (c \cup d) = (b \cup c) \cap (c \cup d) = b \cup c.$$

By (3) we have $a \cup c = b \cup c$. Hence

$$b = (b \cup c) \cap b = (a \cup c) \cap b = (a \cap b) \cup (b \cap c) = (a \cap b) \cup (a \cap c)$$
$$= a \cup (b \cap c) = a \cup (a \cap c) = a.$$

Conversely, suppose $a \cap c = b \cap c$, $a \cup c = b \cup c$ in any lattice L. To complete the proof that L is distributive, we shall shov $a = b$. Let $d = a \cup b$, then $a, b \leqq c \cup d$, and $(a \cup c) \cap d = b \cup c) \cap d$. Thus L satisfies (1) (2) and (3). This shows $a = b$.

Using the notion of p-separatedness, we have already characterized the distributive lattices.

Theorem 5. *The necessary and sufficient condition for a lattice with* 0 *to be distributive is that every pair of distinct elements of* L *be* p*-separable.*

Proof. See K. ISEKI [8].

With the help of Theorem 5, we can now obtain the following

Theorem 6. *A lattice with* 0 *is distributive if and only if every filter is the meet of all prime filters containing it.*

Proof. Let L be a distributive lattice, and F a given filter in L. Suppose $a \in L - F$; by the Lemma 2, there is a meet-irreducible filter M such that $a \notin M$ and $F \subset M$. By Theorem 2 we see that M is a prime filter. This shows that F is the meet of all prime filters containing F. The converse follows easily from theorem 5.

§ 3. Some criteria for Boolean algebras.

Combining well known results, we shall give in this section some conditions for a lattice to be a Boolean algebra.[3])

Theorem 7. *A distributive lattice with* 0 *and* 1 *is a Boolean algebra if and only if every meet-irreducible filter is maximal.*

Proof. If L is a Boolean algebra, then by Theorem 2. every meet-irreducible filter is prime. On the other hand, in any Boolean algebra the prime filters are maximal. Conversely, by Lemma 1., the prime filters are meet-irreducible in any lattice. Therefore if every meet-irreducible filter is maximal, all prime filters are maximal. By a theorem of L. Nachbin [1] and L. Rieger [1], if L is distributive, it is a Boolean algebra.

Following S. Pankajam [1], we define the product complement and sum complement of an element in a lattice as follows.

Definition 7. *The product complement of an element a in a lattice with* 0 *is defined as the element a' for which*

$$a \cap a' = 0$$

holds, and for every x, $a \cap x = 0$ implies $x \leqq a'$.

Definition 8. *The sum-complement of a in a lattice with* 1 *is defined as the element a^* for which $a \cup a^* = 1$, and for every x, $a \cup x = 1$ implies $a^* \leqq x$.*

The product complement and the sum complement are necessarily unique if they exist.

Definition 9. *A lattice in which every element has a product complement is called a lattice with product complement.*

Similarly we can define a lattice with sum complement.

Theorem 8. *A necessary and sufficient condition for a lattice L with product complement to be a Boolean algebra is given by the requirement that every element a of L be normal: $a'' = a$, where $a'' = (a')'$.*

Proof. This theorem has been proved in more general form for semilattices. For details, see P. Samuel [1] or K. Iseki [8].

From this, the following theorem can be easily deduced:

Theorem 9. *If a complete lattice satisfying the infinite distributive law:*

$$x \cap (\bigcup_\alpha y_\alpha) = \bigcup_\alpha (x \cap y_\alpha),$$

has the W-property, then it is a Boolean algebra. (See T. Michiura [1].)

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[3]) The theory of Boolean algebras was extensively studied by M. H. Stone [1]. In the sequel, we shall use his terminology.

Proof. Let a' be the join of all x such that $a \cap x = 0$. By the infinite distributivity,

$$a \cap a' = a \cap (\cup x) = \bigcup (a \cap x) = 0,$$

and $a \cap x = 0$ implies $x \leqq a'$. This shows that a' is the product complement of a. Define $a'' = (a')'$, then we have $a'' = a$. Clearly $a'' \geqq a$; now suppose $a'' \neq a$. By the W-property there is an y such that $a'' \cap y \neq 0$, $a \cap y = 0$. Therefore from $y \leqq a'$ we have $0 = a' \cap a'' \geqq a'' \cap y$ which contradicts $a'' \cap y \neq 0$. Hence L is a Boolean algebra.

Corollary. *In a complete lattice, satisfying the most general distributive law*[4]*), the W-property implies isomorphism with the set-algebra.*

The proof. follows by the Theorem 9 and TARSKI's celebrated theorem (see A. TARSKI [1]).

Now consider a congruence relation on a lattice.

Let F be a filter. The elements a, b are said to be congruent with respect to F

$$a \equiv b(F)$$

if there exist n, y in F such that $a \cap x = b \cap y$.

Dually we define the congruence with respect to an ideal I.

Definition 10. *A subset I of L is an ideal in L if and only if*

1) *$a, b \in I$ implies $a \cup b \in I$;*

2) *$a \in I$ and $a \geqq x$ imply $x \in I$.*

The elements a, b are called congruent for I

$$a \approx b(I)$$

if there exist elements x, y in I such that $a \cup x = b \cup y$.

Clearly we have an equivalence relation:

1) $a \equiv a(F)$;	1') $a \approx a(I)$;
2) $a \equiv b(F) \rightarrow b \equiv a(F)$;	2') $a \approx b(I) \rightarrow b \approx a(I)$;
3) $a \equiv b \cdot b \equiv c(F) \rightarrow a \equiv c(F)$;	3') $a \approx b \cdot b \approx c(I) \rightarrow a \approx c(I)$.

This induces a partition of L into disjoint subsets of equivalent elements called residue classes. Following V. S. KRISHNAN, we define the last residue class of a filter or an ideal. The last residue class of a filter F (an ideal I) is defined as the residue class which contains 0 (1) of L, and it is denoted by F^0 (I^1).

In a distributive lattice with 0 and 1, the last residue class of a filter (an ideal) is an ideal (a filter).

Lemma. *For any filter F or ideal I of a distributive lattice with* 0 *and* 1,

$$F^{01} \subset F, \quad I^{10} \subset I.$$

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[4]) See W. SIERPINSKI, Algèbre des ensembles. *Monografie Mat.* **23** (1951), p. 178, or R. VAIDYANATHASWAMY [2].

Proof. See V. S. KRISHNAN [1].

Theorem 10. *For a Boolean algebra,*

$$F^{01} = F, \quad I^{10} = I.$$

Proof. Let a be an element in F. The relation: $a' \cap a = 0 \cap a = 0$ shows that a' is contained in the last residue class of F, F^0. Next we show $a \in F^{01}$. Since F^0 is an ideal, by the Lemma $0 \in F^0$,

$$a' \cup a = 1 \cup 0 = 1$$

and so $a \in F^{01}$.

Conversely, we can give a characteristic property of Boolean algebras which was proved by T. MICHIURA [1]:

Theorem 11. *A necessary and sufficient condition for a distributive lattice with* 0 *and* 1 *to be a Boolean algebra is that* $F^{01} = F$ *be true for every principal filter (or* $I^{10} = I$ *for every principal ideal).*

From this we infer

Corollary 1. *A necessary and sufficient condition for a distributive lattice with* 0 *and* 1 *to be a Boolean algebra is that every filter (or ideal) be the last residue class of its last residue class.*

Corollary 2. *Each filter of a Boolean algebra is the last residue class of one and only one ideal, and its dual.*

Proof. Let F be a filter; as $F^{01} = F$, F is the last residue class of the ideal F^0. Suppose $F = I_1^1 = I_2^1$, where I_1, I_2 are ideals.

$$I_1 = I_1^{10} = F^0 = I_2^{10} = I_2.$$

This completes the proof.

I state here an unsolved problem: Is any distributive lattice with 0 and 1, each ideal of which is the last residue class of one and only filter, necessarily a Boolean algebra?

The concept of BROUWERIAN algebra was introduced by A. TARSKI and J. C. C. MCKINSEY. This structure is defined to be a lattice L with 0, satisfying the following axioms:

1. L is closed under a binary operation $\dot{-}$.
2. $a \dot{-} b \leq c$ and $b \cdot \leq a \cup c$ are equivalent for $a, b, c \in L$.

$\urcorner a = 1 \dot{-} a$ is called the Brouwerian complement of a.

Lemma. *Any Brouwerian algebra is a distributive lattice with sum-complement.*

Proof. It is known that such a Brouwerian algebra is a distributive lattice. $\urcorner a = 1 \dot{-} a$ means by the axiom 2) $\urcorner a \cup a = 1$. Similarly $a \cup b = 1$ means $\urcorner a = 1 \dot{-} a \leq b$. Hence $\urcorner a$ is the sum complement of a.

By the Lemma, we have

Theorem 12. *A necessary and sufficient condition for a Brouwerian algebra with* 0 *to be a Boolean algebra is that, for every a in L,*

$$\urcorner a \cap a = 0.$$

§ 4. Atomic lattices with W-property.

An element which covers 0 in a lattice with 0 is called an atom. A lattice L will be called atomic if every non-zero element in L contains at least one atom. The definition of atomic lattice is found in O. FRINK [1].

Following O. FRINK [1], we shall define the representation set of an element a in the atomic lattice L. By $r(a)$, we mean the set of all atoms x of L such that $x \leqq a$.

Theorem 13. *An element a of an atomic lattice with W-property is the join of all elements in $r(a)$: $a = \bigcup_{x \in r(a)} x$.*

Proof. For $a = 0$, the theorem is obvious. Suppose $a \neq 0$: for $x \in r(a)$ we have $x \leqq a$. This means that a is an upper bound of elements in $r(a)$. Let b be an upper bound of elements in $r(a)$ such that $b < a$. Since L has the W-property, there is a non-zero element c such that $b \cap c = 0$, $c \leqq a$. Since L is atomic, there exists an atomic element x such that $x \leqq a$. Therefore $r(a)$ must contain x: $x \in r(a)$. By the property of b, $b \geqq x$, this however is a contradiction. a must be the join of the elements of $r(a)$.

Theorem 14. *An atomic lattice L with W-property is isomorphic with the lattice of all represantative sets of L.*

Proof. It is sufficient to show the following three properties:

(1) $r(a) \cap r(b) = r(a \cap b)$

(2) $r(a) \cup r(b) = r(a \cup b)$

(3) $a \neq b \rightarrow r(a) \neq r(b)$.

Let x be an element of $r(a \cap b)$, then $a \cap b \geqq x$. Therefore $x \leqq a, b$. This shows $x \in r(a) \cap r(b)$. Hence $r(a \cap b) < r(a) \cap r(b)$. Since $x \in r(a) \cap r(b)$, we have $x \leqq a, b$. Hence $x \in r(a \cup b)$. This completes the proof of the relation (1). The relation $r(a \cup b) > r(a) \cup r(b)$ is obvious. We shall show that $r(a \cup b)$ is the join of $r(a)$ and $r(b)$. Let $r(c)$ be a representative set such that $r(c) > r(a), r(b)$ and $r(a \cup b) \not\subseteq r(c)$. Then there is an atom x such that $x \leqq a \cup b$ and $x \not\leqq c$. By Theorem 13 we have $a = \bigcup_{y \in r(a)} y \leqq \bigcup_{z \in r(c)} z = c$ and $b \leqq c$. This shows $x \leqq a \cup b \leqq c$ which is a contradiction. To show the implication (3) we prove that $r(a) = r(b)$ implies $a = b$. This follows immediately from Theorem 13.

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