On the conjugate mapping for quaternions.

By K. Iseki in Osaka (Japan).

In this note we give two axiomatic definitions of the conjugate mapping in the quaternion skew field over the real field.

We shall use the symbols $1, i, j, k$ to denote the base of the quaternions which satisfies the following multiplication relations:

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Therefore a quaternion x may be represented in the form $x = x_1 + x_2 i + x_3 j + x_4 k$ with real coefficients x_i $(i = 1, 2, 3, 4)$. By the conjugate number $\bar{x}$ of x, we shall mean $x = x_1 - x_2 i - x_3 j - x_4 k$. Under the norm $\|x\|$ of x we shall understand $\|x\| = +\sqrt{x\bar{x}} = +\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$. Then the norm so defined satisfies the condition $\|xy\| = \|x\| \|y\|$. Let $f(x)$ be a mapping of the quaternion field into itself. We shall prove the following theorems which give the necessary and sufficient conditions for $f(x)$ to be the conjugate of x.

Theorem 1. *$f(x)$ is the conjugate of x if and only if it satisfies the conditions:*

(1) *$f(x)$ is continuous at zero (with respect to the norm),*

(2) $f(x+y) = f(x) + f(y)$,

(3) *$f(x) = x$ for every real x,*

(4) *$f(i) = -i$, $f(j) = -j$, and $f(k) = -k$.*

Theorem 2. *$f(x)$ is the conjugate of x if and only if it satisfies the conditions:*

(1) *$f(x)$ is continuous at zero,*

(2) $xf(x) = f(x)x$,

(3) *$xf(x)$ is real for every x,*

(4) *$x + f(x)$ is real for every x.*

Analogous theorems for complex numbers have been obtained by St. Golab[1]).

If $f(x) = \bar{x}$, then $f(x)$ satisfies obviously the conditions of the theorems.

Proof of theorem 1. $f(x)$ may be expressed in the form $\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ with real functions $\alpha_i = \alpha_i(x_1, x_2, x_3, x_4)$, $(i = 1, 2, 3, 4)$, where $x = x_1 +$

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[1]) St. Golab, Sur une définition axiomatique des nombres conjugués pour les nombres complexes ordianires. *Opuscula Math.* **1** (1937), pp. 1-11.

$+x_2 i+x_3 j+x_4 k$. From $f(x+y)=f(x)+f(y)$, we have

$$\alpha_1(x_1, x_2, x_3, x_4)+\alpha_2(x_1, x_2, x_3, x_4)i+\alpha_3(x_1, x_2, x_3, x_4)j+\alpha_4(x_1, x_2, x_3, x_4)k+$$
$$+\alpha_1(y_1, y_2, y_3, y_4)+\alpha_2(y_1, y_2, y_3, y_4)i+\alpha_3(y_1, y_2, y_3, y_4)j+\alpha_4(y_1, y_2, y_3, y_4)k=$$
$$=\alpha_1(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4)+\alpha_2(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4)i+$$
$$+\alpha_3(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4)j+\alpha_4(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4)k.$$

This implies that the α_i $(i=1, 2, 3, 4)$ are linear:

$$\alpha_i(x_1, x_2, x_3, x_4)+\alpha_i(y_1, y_2, y_3, y_4)=\alpha_i(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4).$$

Moreover, condition (3) implies

$$\alpha_1(x_1, 0, 0, 0)=x_1,$$
$$\alpha_i(x_1, 0, 0, 0)=0 \quad (i=2, 3, 4).$$

Let $\beta_i(x_2)=\alpha_i(0, x_2, 0, 0)$, $(i=1, 2, 3, 4)$. Then $\beta_i(x_2+y_2)=\beta_i(x_2)+\beta_i(y_2)$ $(i=1, 2, 3, 4)$. From this we infer by the continuity of $\beta_i(x_2)$ at zero, that $\beta_i(x_2)=c_i x_2$ where the c_i's are real constants. Putting $x=i$, we have by condition (4) $-i=f(i)=\beta_1(0)+\beta_2(1)i+\beta_3(0)j+\beta_4(0)k=c_2 i$. This yields $c_2=-1$. Similarly we have $c_3=c_4=-1$. Thus we see that $f(x)=x_1-x_2 i-x_3 j-x_4 k=\bar{x}$ which completes the proof.

Proof of theorem 2. With the notation previously used we have

$$\begin{aligned} xf(x)=&(x_1\alpha_1-x_2\alpha_2-x_3\alpha_3-x_4\alpha_4)\\ &+(x_2\alpha_1+x_1\alpha_2-x_4\alpha_3+x_3\alpha_4)i\\ &+(x_3\alpha_1+x_4\alpha_2+x_1\alpha_3-x_2\alpha_4)j\\ &+(x_4\alpha_1-x_3\alpha_2+x_2\alpha_3+x_1\alpha_4)k.\\ f(x)x=&(x_1\alpha_1-x_2\alpha_2-x_3\alpha_3-x_4\alpha_4)\\ &+(x_1\alpha_2+x_2\alpha_1-x_3\alpha_4+x_4\alpha_3)i\\ &+(x_1\alpha_3+x_2\alpha_4+x_3\alpha_1-x_4\alpha_2)j\\ &+(x_1\alpha_4-x_2\alpha_3+x_3\alpha_2-x_4\alpha_1)k.\end{aligned}$$

By condition (2) we have

$$x_3\alpha_4=x_4\alpha_3, \quad x_2\alpha_4=x_4\alpha_2, \quad x_2\alpha_3=x_3\alpha_2.$$

Condition (3) implies

$$x_2\alpha_1+x_1\alpha_2-x_4\alpha_3+x_3\alpha_4=0,$$
$$x_3\alpha_1+x_4\alpha_2+x_1\alpha_3-x_2\alpha_4=0,$$
$$x_4\alpha_1-x_3\alpha_2+x_2\alpha_3+x_1\alpha_4=0,$$

i. e. according to our above equations,

$$x_2\alpha_1+x_1\alpha_2=0, \quad x_3\alpha_1+x_1\alpha_3=0, \quad x_4\alpha_1+x_1\alpha_4=0.$$

Condition (4) means that $x_2+\alpha_2=0$, $x_3+\alpha_3=0$, $x_4+\alpha_4=0$. Consequently $x_2(x_1-\alpha_1)=0$. Then, for $x_2\neq 0$, we have $\alpha_1=x_1$. Hence $f(x)=x_1-x_2 i-x_3 j-x_4 k$ for $x_2\neq 0$. By the continuity of $f(x)$, $f(x)=\bar{x}$ for all quaternions x. This completes the proof.

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