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On generalized rectangular and rhombic functional equations

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1. Introduction

In [2], the functional equations

$$\text{(RE)}\qquad \begin{aligned} &f(x_1+y_1,x_2+y_2)+f(x_1+y_1,x_2-y_2)+\\ &+f(x_1-y_1,x_2+y_2)+f(x_1-y_1,x_2-y_2)=4\,f(x_1,x_2)\end{aligned}$$

and

$$\text{(RH)}\qquad \begin{aligned} &f(x_1+y_1,x_2)+f(x_1-y_1,x_2)+\\ &+f(x_1,x_2+y_2)+f(x_1,x_2-y_2)=4\,f(x_1,x_2),\end{aligned}$$

among others, were considered for f mapping $\mathfrak{R}^2$ into $\mathfrak{R}$ (the reals). For obvious geometric reasons, these equations are referred to as the rectangular and rhombic equations, respectively. Their general solutions are the same: $f(x_1,x_2)=A(x_1,x_2)+B(x_1)+C(x_2)+\alpha$, where A is an arbitrary biadditive map, B and C are arbitrary additive maps, and α is an arbitrary constant.

In the present paper, we generalize those results in three ways. For one thing, we consider a more general right hand side. Also, we generalize the domain to a product of groups and the range to a field. And thirdly, we deal with functions of any finite number of variables. Specifically, we consider the equations

$$\text{(GRE)}\qquad \begin{aligned} &\sum_{\sigma_1,\dots,\sigma_n=\pm1} f(x_1y_1^{\sigma_1},\dots,x_ny_n^{\sigma_n})=\\ &=f(x_1,\dots,x_n)\,g(y_1,\dots,y_n)+h(y_1,\dots,y_n)\end{aligned}$$

and

$$\text{(GRH)} \qquad \sum_{i=1}^{n} \sum_{\sigma_i=\pm 1} f(x_1, ..., x_{i-1}, x_i y_i^{\sigma_i}, x_{i+1}, ..., x_n) = \\ = f(x_1, ..., x_n)\, p(y_1, ..., y_n) + q(y_1, ..., y_n)$$

for $f, g, h, p, q : \mathbf{G}_1 \times \mathbf{G}_2 \times \cdots \times \mathbf{G}_n \to \mathbf{K}$, where each $\mathbf{G}_i$ $(i = 1, 2..., n)$ is a group and $\mathbf{K}$ is a quadratically closed (commutative) field of characteristic different from 2.

In Section 3 of the paper, we find the general solution of the generalized rectangular equation (GRE). In Section 4, we show that the class of all f satisfying the generalized rhombic equation (GRH) is identical with the class of all f satisfying a generalized rectangular equation. Then the result of Section 3 is invoked to obtain the general solution of the generalized rhombic equation. In Section 5, we complete the link between (GRE) and (GRH) and give the regular solutions of these equations.

2. Some preliminaries

The proof of the main results are by induction on n, the number of variables. Hence our starting point is provided by the following result, which covers the initial step $n = 1$.

Lemma 1. *The general solution* $f, g, h, : \mathbf{G} \to \mathbf{K}$ *of*

$$\text{(2.1)} \qquad f(xy) + f(xy^{-1}) = f(x)\, g(y) + h(y)$$

with f *satisfying the factorization condition*

$$\text{(FC)} \qquad f(xyz) = f(xzy)$$

is provided by

$$\text{(2.2)} \qquad f(x) = \gamma, \quad g \text{ arbitrary}, \quad h(y) = \gamma[2 - g(y)];$$

$$\text{(2.3)} \qquad f(x) = Q(x) + A(x) + \gamma, \quad g(y) = 2, \quad h(y) = 2Q(y);$$

$$\text{(2.4)} \qquad f(x) = \alpha\, \psi(x) + \beta\, \psi(x)^{-1} + \gamma, \quad g(y) = \psi(y) + \psi(y)^{-1} \\ h(y) = \gamma\, \{2 - [\psi(y) + \psi(y)^{-1}]\}, \qquad \psi(y) \not\equiv \psi(y)^{-1};$$

or by

$$\text{(2.5)} \qquad f(x) = [A(x) + \alpha]\psi(x) + \gamma, \quad g(y) = 2\, \psi(y) \\ h(y) = 2\, \gamma\, [1 - \psi(y)], \qquad \psi(y) \equiv \psi(y)^{-1} \not\equiv 1,$$

where $\alpha, \beta, \gamma \in \mathbf{K}$ *are arbitrary constants,* $Q : \mathbf{G} \to \mathbf{K}$ *is a quadratic function:*

$$Q(xy) + Q(xy^{-1}) = 2Q(x) + 2Q(y),$$

$A : \mathbf{G} \to \mathbf{K}$ *is an additive function:*

$$A(xy) = A(x) + A(y),$$

and $\psi : \mathbf{G} \to \mathbf{K}$ *is a nonzero exponential function:*

$$\psi(xy) = \psi(x)\,\psi(y), \qquad \psi \not\equiv 0.$$

Proof. Equation (2.1) is a special case of an equation treated in Theorem 3.1 in [3]. An exhaustive case-by-case reduction of the solutions provided there leads to (2.2)–(2.5).

Remark 1. The condition (FC) essentially allows us to operate as if the function is defined on the abelian group $\mathbf{G}/\mathbf{C}$, where $\mathbf{C}$ is the commutator subgroup of $\mathbf{G}$ (see [4, p. 136]).

Remark 2. Any such quadratic function Q as above can be described as the diagonal of a biadditive function $A_{12} : \mathbf{G} \times \mathbf{G} \to \mathbf{K}$ (see [1]). We choose the quadratic nomenclature and notation here to avoid confusion with the many other multiadditive functions which will appear.

In the next section, we shall solve the generalized rectangular equation (GRE) under the supposition that f satisfies the factorization condition (FC) in each variable. In describing the solutions of (GRE), the following definition will be helpful. Any function $S : \mathbf{G}_1 \times \mathbf{G}_2 \times \cdots \times \mathbf{G}_n \to \mathbf{K}$ of the form

$$\begin{aligned} S(x_1, x_2, ..., x_n) = {} & A_{12\cdots n}(x_1, x_2, ..., x_n) + \\ & + \left[A_{12\cdots(n-1)}(x_1, x_2, ..., x_{n-1}) + \cdots + \right. \\ & \left. + A_{23\cdots n}(x_2, x_3, ..., x_n)\right] + \\ & + \cdots + \left[A_1(x_1) + \cdots + A_n(x_n)\right], \end{aligned}$$

where each $A_{j_1\cdots j_k} : \mathbf{G}_{j_1} \times \cdots \times \mathbf{G}_{j_k} \to \mathbf{K}$ $(1 \leq j_1 < \cdots < j_k \leq n)$ is additive in each variable, is termed a *sum of multiadditive functions* (SMAF) of order n. For example, any function of the form $A_{12}(x_1, x_2) + A_1(x_1) + A_2(x_2) + \alpha$, where A_{12} is biadditive and A_1 and A_2 are additive, could be written as $S(x_1, x_2) + \alpha$, where S is a SMAF of order 2.

3. General solution of (GRE)

Now we are ready to establish the first main result.

Theorem 1. *The general solution* $f, g, h : \mathbf{G}_1 \times \cdots \times \mathbf{G}_n \to \mathbf{K}$ *of* (GRE) *with* f *satisfying* (FC) *in each variable, is given by (with* $X = (x_1, x_2, ..., x_n)$, $Y = (y_1, y_2, ..., y_n)$*)*

$$f(X) = \gamma, \quad g \text{ arbitrary}, \quad h(Y) = \gamma\,[2^n - g(Y)], \tag{3.1}$$

$$f(X) = S(X) + \alpha + \sum_{i=1}^{n} Q_i(x_i), \tag{3.2}$$

$$g(Y) = 2^n, \qquad h(Y) = 2^n \sum_{i=1}^{n} Q_i(y_i),$$

$$\text{(3.3a)} \quad \begin{cases} f(X) = \displaystyle\sum_{\sigma_1,...,\sigma_k=\pm 1} \big[S_{\sigma_1,...,\sigma_k}(x_{p_1}, ..., x_{p_{n-k}}) \\ \qquad\qquad + \alpha_{\sigma_1,...,\sigma_k}\big] \displaystyle\prod_{i=1}^{n-k} \psi_{p_i}(x_{p_i}) \prod_{j=1}^{k} \psi_{r_j}(x_{r_j})^{\sigma_j} + \gamma \\ g(Y) = \displaystyle\prod_{i=1}^{n} \left[\psi_i(y_i) + \psi_i(y_i)^{-1}\right], \quad h(Y) = \gamma\,[2^n - g(Y)] \end{cases}$$

$$\text{(3.3b)} \quad \begin{cases} f(X) = [S(x_1, ..., x_n) + \alpha] \displaystyle\prod_{i=1}^{n} \psi_i(x_i) + \gamma, \\ \qquad \psi_i(\cdot) = \psi_i(\cdot)^{-1}, \ i = 1, 2, ..., n, \\ g(Y) = 2^n \displaystyle\prod_{i=1}^{n} \psi_i(y_i) \not\equiv 2^n, \quad h(Y) = \gamma\,[2^n - g(Y)], \end{cases}$$

where k *is a fixed integer* $(1 \le k \le n)$, α, γ *and* $\alpha_{\sigma_1,...,\sigma_k}$ $(\sigma_1, ..., \sigma_k = \pm 1)$ *are arbitrary constants,* S *and* $S_{\sigma_1,...,\sigma_k}$ $(\sigma_1, ..., \sigma_k = \pm 1)$ *are* SMAF's *of order* n *and* $n-k$*, respectively,* Q_i *is quadratic and* ψ_i *is a nonzero exponential* $(i = 1, 2, ..., k)$, ψ_{p_i} $(i = 1, 2, ..., n-k)$ *satisfies also* $\psi_{p_i}(\cdot) = \psi_{p_i}(\cdot)^{-1}$, ψ_{r_j} $(j = 1, 2, ..., k)$ *satisfies also* $\psi_{r_j}(\cdot) \neq \psi_{r_j}(\cdot)^{-1}$, *and* $\{1, 2, ..., n\}$ *is the (disjoint) union of the sets* $\{p_1, ..., p_{n-k}\}$ *and* $\{r_1, ..., r_k\}$.

Here we have adopted the convention that in (3.3a)

$$S_{\sigma_1,\dots,\sigma_k}(x_{p_1},...,x_{p_{n-k}}) := 0$$

(C1)
$$\text{and} \quad \prod_{i=1}^{n-k}\psi_{p_i}(x_{p_i}) := 1 \qquad \text{if} \quad k=n.$$

Remark 3. Note that (3.3b) can be included as the special case $k=0$ of (3.3a) if we interpret $\sum_{\sigma_1,\dots,\sigma_k=\pm1}$ as a single term and $\prod_{j=1}^{k}=1$ when $k=0$. The statement and proof of the theorem seem clearer, however, if we separate this case.

Proof. The proof is by induction on n. For $n=1$, it is Lemma 1. Suppose now that the statement is true for $n=N\geq 1$, and consider (GRE) for $n=N+1$.

If $f=\gamma$, then (GRE) yields $2^n\gamma=\gamma g(Y)+h(Y)$. This gives solution (3.1) immediately, and henceforth we assume that f is nonconstant.

Putting $y_i=e_i$ (the identity of $\mathbf{G}_i$) for $i=1,2,...,N$ in (GRE), we get

$$2^N[f(x_1,...,x_N,x_{N+1}y_{N+1})+f(x_1,...,x_N,x_{N+1}y_{N+1}^{-1})]=$$
$$=f(x_1,...,x_{N+1})g(e_1,...,e_N,y_{N+1})+h(e_1,...,e_N,y_{N+1}).$$

Since char $\mathbf{K}\neq 2$, this can be written as

$$f(x_1,...,x_N,x_{N+1}y_{N+1})+f(x_1,...,x_N,x_{N+1}y_{N+1}^{-1})=$$
$$=f(x_1,...,x_{N+1})g'(y_{N+1})+h'(y_{N+1}).$$

For each fixed $(x_1,...,x_N)\in\mathbf{G}_1\times\cdots\times\mathbf{G}_N$, this is an equation of the form (2.1). By Lemma 1, f must have one of the following three forms, with Q_{N+1} quadratic, A_{N+1} additive in its last variable, and ψ_{N+1} a nonzero exponential:

(3.4)
$$f(x_1,...,x_{N+1})=$$
$$=Q_{N+1}(x_{N+1})+A_{N+1}(x_1,...,x_{N+1})+B(x_1,...,x_N),$$

(3.5)
$$f(x_1,...,x_{N+1})=A(x_1,...,x_N)\,\psi_{N+1}(x_{N+1})+$$
$$+B(x_1,...,x_N)\,\psi_{N+1}(x_{N+1})^{-1}+\gamma \quad \text{with} \quad \psi_{N+1}(x)\not\equiv\psi_{N+1}(x)^{-1},$$

(3.6)
$$f(x_1,...,x_{N+1})=[A_{N+1}(x_1,...,x_{N+1})+$$
$$+B(x_1,...,x_N)]\,\psi_{N+1}(x_{N+1})+\gamma \quad \text{with} \quad \psi_{N+1}(x)\equiv\psi_{N+1}(x)^{-1}\not\equiv 1.$$

In writing (3.5), we have used implicitly the fact that nonzero exponentials are independent if distinct.

We consider each of these cases in turn.

Case 1. Substituting the form (3.4) of f into (GRE), we simplify using the fact that Q_{N+1} is quadratic and A_{N+1} is additive in its last variable. Thus we arrive at

$$(3.7)\qquad 2^{N+1}\left[Q_{N+1}(x_{N+1})+Q_{N+1}(y_{N+1})\right]+$$
$$+2\sum_{\sigma_1,\ldots,\sigma_N=\pm 1}\left[A_{N+1}(x_1y_1^{\sigma_1},\ldots,x_Ny_N^{\sigma_N},x_{N+1})+B(x_1y_1^{\sigma_1},\ldots,x_Ny_N^{\sigma_N})\right]$$
$$=\left[Q_{N+1}(x_{N+1})+A_{N+1}(x_1,\ldots,x_{N+1})+B(x_1,\ldots,x_N)\right]g(Y)+h(Y).$$

Since $Q_{N+1}(\cdot)$, $A_{N+1}(x_1,\ldots,x_N,\cdot)$ and 1 are linearly independent (if nonzero), we deduce that

$$(3.7a)\qquad 2^{N+1}Q_{N+1}(x_{N+1})=Q_{N+1}(x_{N+1})\,g(Y),$$

$$(3.7b)\qquad \sum_{\sigma_1,\ldots,\sigma_N=\pm 1}A_{N+1}(x_1y_1^{\sigma_1},\ldots,x_Ny_N^{\sigma_N},x_{N+1})=$$
$$=A_{N+1}(x_1,\ldots,x_{N+1})\,\frac{1}{2}\,g(Y),$$

$$(3.7c)\qquad \sum_{\sigma_1,\ldots,\sigma_N=\pm 1}B(x_1y_1^{\sigma_1},\ldots,x_Ny_N^{\sigma_N})=$$
$$=B(x_1,\ldots,x_N)\,\frac{1}{2}\,g(Y)+\frac{1}{2}\,\left[h(Y)-2^{N+1}\,Q_{N+1}(y_{N+1})\right].$$

Now let us consider three subcases. First, suppose that $g(Y)\equiv 2^{N+1}$. Then (3.7a,b,c) yields

$$(3.8a)\qquad \sum_{\sigma_1,\ldots,\sigma_N=\pm 1}A_{N+1}(x_1y_1^{\sigma_1},\ldots,x_Ny_N^{\sigma_N},x_{N+1})=$$
$$=2^N\,A_{N+1}(x_1,\ldots,x_{N+1}),$$

$$(3.8b)\qquad \sum_{\sigma_1,\ldots,\sigma_N=\pm 1}B(x_1y_1^{\sigma_1},\ldots,x_Ny_N^{\sigma_N})=$$
$$=2^N\,B(x_1,\ldots,x_N)+\frac{1}{2}\,[h(Y)-2^{N+1}Q_{N+1}(y_{N+1})].$$

Equations (3.8a) and (3.8b) are of the form (GRE), to which we can apply the induction hypothesis. Note that with the particular form of g here, (3.1) is a special case of (3.2), while (3.3a) and (3.3b) are not possible. (In the case of (3.3a), $\prod_{i=1}^{N}[\psi_i(y_i)+\psi_i(y_i)^{-1}] \equiv 2^N$ is possible only if $\psi_i = 1$ for each i. But this is impossible, since $k \geq 1$.) Hence we obtain from (3.8a) and (3.8b) that

$$\text{(3.9a)} \qquad A_{N+1}(x_1,...,x_N,x_{N+1}) = S_A(x_1,...,x_N;x_{N+1}) + C_A(x_{N+1}),$$

$$\text{(3.9b)} \qquad B(x_1,...,x_N) = S_B(x_1,...,x_N) + \alpha + \sum_{i=1}^{N} Q_i(x_i),$$

$$\text{(3.9c)} \qquad \frac{1}{2}\,h(Y) - 2^N Q_{N+1}(y_{N+1}) = 2^N \sum_{i=1}^{N} Q_i(y_i),$$

where S_A is a SMAF of order N in its first N variables, S_B is a SMAF of order N, and Q_i $(i = 1, 2, ..., N)$ is quadratic.

Moreover, since A_{N+1} is additive in its last variable, (3.9a) and the linear independence of $S_A(x_1,...,x_{N+1})$ and $C_A(x_{N+1})$ as functions of $x_1,..,x_N$ (if nonzero) show that S_A is additive in its last variable and that C_A is additive. Hence, the map $S : \mathbf{G}_1 \times \cdots \times \mathbf{G}_{N+1} \to \mathbf{K}$ defined by

$$S(x_1,...,x_{N+1}) := S_A(x_1,...,x_N;x_{N+1}) + C_A(x_{N+1}) + S_B(x_1,...,x_N)$$

is a SMAF of order $N + 1$. Together with (3.9a,b,c) and (3.4), this shows that in this case we have a solution of the form (3.2) for $n = N + 1$.

Next, suppose that $g(Y) \not\equiv 2^{N+1}$ and $A_{N+1} = 0$. Then by (3.7a) we have

$$\text{(3.10)} \qquad Q_{N+1} = 0.$$

Now (3.4) reduces to $f(x_1,...,x_{N+1}) = B(x_1,...,x_N)$. Moreover, (3.7c) (now with Q_{N+1}=0) is, for each fixed y_{N+1}, of the form (GRE) with n=N. Applying the induction hypothesis, since B is nonconstant, we conclude that B, $\frac{1}{2}g$, $\frac{1}{2}h$ have the forms of f, g, h (respectively) in either (3.2), (3.3a) or (3.3b) for each y_{N+1}. The first case, (3.2), is impossible as $\frac{1}{2}g(Y) \equiv 2^N$ contradicts the hypothesis $g(Y) \not\equiv 2^{N+1}$. In the second case, the solution is of the form (3.3a) for $n = N+1$, with $\psi_{N+1} = 1$, where $p_{N+1-k} := N+1$ and each $S_{\sigma_1,...,\sigma_k}(x_{p_1},...,x_{p_{N+1-k}}) := S_{\sigma_1,...,\sigma_k}(x_{p_1},...,x_{p_{N-k}})$. Similarly, in the third case the solution is of the form (3.3b).

Finally, suppose that $g(Y) \not\equiv 2^{N+1}$ and $A_{N+1} \neq 0$. We again get (3.10) from (3.7a). Furthermore, (3.7b) shows that $g(Y)$ is independent of

y_{N+1}. Thus, for each fixed x_{N+1}, (3.7b) is of the form (GRE) for $n = N$ with $h = 0$. By induction hypothesis, we deduce that $A_{N+1}(x_1, ..., x_N, x_{N+1})$ must be of the form (3.3a) or (3.3b) in the first N variables, for each fixed x_{N+1}. We also find that

$$\frac{1}{2}\, g(Y) = \prod_{i=1}^{N} \left[\psi_i(y_i) + \psi_i(y_i)^{-1}\right] \not\equiv 2^N \tag{3.11}$$

and that $\gamma = 0$ in (3.3a), (3.3b). That is, A_{N+1} is given by

$$\begin{aligned} A_{N+1}(X) = \sum_{\sigma_1,...,\sigma_k=\pm 1} & \Big[S^{(A)}_{\sigma_1,...,\sigma_k}(x_{p_1}, ..., x_{p_{N-k}}, x_{N+1}) + \\ & + \alpha^{(A)}_{\sigma_1,...,\sigma_k}(x_{N+1})\Big] \prod_{i=1}^{N-k} \psi_{p_i}(x_{p_i}) \prod_{j=1}^{k} \psi_{r_j}(x_{r_j})^{\sigma_j} \end{aligned} \tag{3.12a}$$

or

$$\begin{aligned} A_{N+1}(X) = & \Big[S^{(A)}(x_1, ..., x_N, x_{N+1}) + \\ & + \alpha^{(A)}(x_{N+1})\Big] \prod_{i=1}^{N} \psi_i(x_i), \qquad \psi_i(\cdot) = \psi_i(\cdot)^{-1},\ i = 1, 2, ..., N, \end{aligned} \tag{3.12b}$$

where $S^{(A)}$ and $S^{(A)}_{\sigma_1,...,\sigma_k}$ are SMAF's of order N and $N-k$ (respectively) in all variables but the last, and where $S^{(A)}$, $S^{(A)}_{\sigma_1,...,\sigma_k}$, $\alpha^{(A)}$ and $\alpha^{(A)}_{\sigma_1,...,\sigma_k}$ are additive functions of x_{N+1}.

Using (3.10) and (3.11), we obtain

$$\begin{aligned} & \sum_{\sigma_1,...,\sigma_N=\pm 1} B(x_1 y_1^{\sigma_1}, ..., x_N y_N^{\sigma_N}) = \\ & = B(x_1, ..., x_N) \prod_{i=1}^{N} \left[\psi_i(y_i) + \psi_i(y_i)^{-1}\right] + \frac{1}{2} h(Y) \end{aligned} \tag{3.13}$$

from (3.7c). This shows that $h(Y)$, is independent of y_{N+1}. Applying the induction hypothesis, we see that B is given by (3.3a) or (3.3b). Moreover, the form of B corresponds to the form of A_{N+1} given in (3.12a) or (3.12b), according to the set of indices $\{p_1, ..., p_{N-k}\}$ for which $\psi_{p_i}(x) \equiv \psi_{p_i}(x_i)^{-1}$.

If A_{N+1} is given by (3.12a) then B and h are given by

$$B(X) = \sum_{\sigma_1,\ldots,\sigma_k=\pm 1} \left[S^{(B)}_{\sigma_1,\ldots,\sigma_k}(x_{p_1},\ldots,x_{p_{N-k}}) + \alpha_{\sigma_1,\ldots,\sigma_k}\right] \times$$

$$\times \prod_{i=1}^{N-k} \psi_{p_i}(x_{p_i}) \prod_{j=1}^{k} \psi_{r_j}(x_{r_j})^{\sigma_j} + \gamma,$$

$$\frac{1}{2}\, h(Y) = \gamma \left[2^N - g(Y)\right].$$

In this case, defining $p_{N-k+1} := N+1$, $\psi_{N+1} = 1$, and

$$S_{\sigma_1,\ldots,\sigma_k}(x_{p_1},\ldots,x_{p_{N+1-k}}) := S^{(A)}_{\sigma_1,\ldots,\sigma_k}(x_{p_1},\ldots,x_{p_{N-k}}; x_{N+1}) +$$

$$+\alpha^{(A)}_{\sigma_1,\ldots,\sigma_k}(x_{N+1}) + S^{(B)}_{\sigma_1,\ldots,\sigma_k}(x_{p_1},\ldots,x_{p_{N-k}}),$$

we deduce from (3.4) that f has the form given by in (3.3a), with $n = N+1$. We see that g and h are as given in (3.3a) also.

The other possibility is that A_{N+1} is given by (3.12b). In the same manner as the previous case, we find that the solution is of the form (3.3b) for $n = N+1$.

Case 2. Substituting (3.5) for f into (GRE) and simplifying, using the fact that ψ_{N+1} is nonzero exponential, we find that

$$\begin{aligned}
&\sum_{\sigma_1,\ldots,\sigma_N=\pm 1} [A(x_1 y_1^{\sigma_1},\ldots,x_N y_N^{\sigma_N})\, \psi_{N+1}(x_{N+1}) \\
&+B(x_1 y_1^{\sigma_1},\ldots,x_N y_N^{\sigma_N})\, \psi_{N+1}(x_{N+1})^{-1}]\, [\psi_{N+1}(y_{N+1}) + \\
&+\psi_{N+1}(y_{N+1})^{-1}] + 2^{N+1}\gamma = [A(x_1,\ldots,x_N)\, \psi_{N+1}(x_{N+1}) + \\
&+B(x_1,\ldots,x_N)\, \psi_{N+1}(x_{N+1})^{-1} + \gamma]\, g(Y) + h(Y).
\end{aligned} \tag{3.14}$$

Since $\psi_{N+1}(x) \not\equiv \psi_{N+1}(x)^{-1}$, it follows that $\psi_{N+1}(\cdot)$, $\psi_{N+1}(\cdot)^{-1}$ and 1 are linearly independent. Thus, we deduce from (3.14) that

$$\begin{aligned}
&\sum_{\sigma_1,\ldots,\sigma_N=\pm 1} A(x_1 y_1^{\sigma_1},\ldots,x_N y_N^{\sigma_N}) \times \\
&\times[\psi_{N+1}(y_{N+1}) + \psi_{N+1}(y_{N+1})^{-1}] = A(x_1,\ldots,x_N)\, g(Y)
\end{aligned} \tag{3.15}$$

(3.16)
$$\sum_{\sigma_1,\ldots,\sigma_N=\pm 1} B(x_1 y_1^{\sigma_1},\ldots,x_N y_N^{\sigma_N})\times$$
$$\times[\psi_{N+1}(y_{N+1})+\psi_{N+1}(y_{N+1})^{-1}] = B(x_1,\ldots,x_N)\, g(Y)$$

(3.17)
$$2^{N+1}\gamma = \gamma\, g(Y) + h(Y).$$

Furthermore, as we have supposed that f is nonconstant, (3.5) shows that at least one of A or B is nonzero. Suppose, without loss of generality, that $A \neq 0$. Then (3.15) (with $y_{N+1} = e_{N+1}$) yields

(3.18a)
$$\sum_{\sigma_1,\ldots,\sigma_N=\pm 1} A(x_1 y_1^{\sigma_1},\ldots,x_N y_N^{\sigma_N}) =$$
$$= A(x_1,\ldots,x_N)\,\frac{1}{2}\, g(y_1,\ldots,y_N,e_{N+1})$$

(3.18b)
$$g(Y) = \frac{1}{2}\, g(y_1,\ldots,y_N,e_{N+1})\,\left[\psi_{N+1}(y_{N+1})+\psi_{N+1}(y_{N+1})^{-1}\right].$$

By the induction hypothesis, all solutions of (3.18a) must be of the form (3.1), (3.2), or (3.3). The solution can be of the form (3.1) with $A \neq 0$ and no h-term only if (cf. (3.15)) $\frac{1}{2}g(y_1,\ldots,y_N,e_{N+1}) \equiv 2^N$. But then by (3.18b) this solution is a special case of (3.3). Similarly, the solution can be of the form (3.2) with $A \neq 0$ and no h-term only if $Q_i = 0$ $(i = 1,2,..,N)$, and again such a solution is a special case of (3.3).

Therefore, the solution of (3.18a) must be of the form (3.3) with no h-term. That is, either

(3.19a)
$$A(x_1,\ldots,x_N) =$$
$$= \sum_{\sigma_1,\ldots,\sigma_k=\pm 1}\left[S^{(A)}_{\sigma_1,\ldots,\sigma_k}(x_{p_1},\ldots,x_{p_{N-k}})+\alpha^{(A)}_{\sigma_1,\ldots,\sigma_k}\right]\times$$
$$\times\prod_{i=1}^{N-k}\psi_{p_i}(x_{p_i})\prod_{j=1}^{k}\psi_{r_j}(x_{r_j})^{\sigma_j}+\gamma,$$

or

(3.19b)
$$A(x_1,\ldots,x_N) =$$
$$= \left[S^{(A)}(x_1,\ldots,x_N)+\alpha^{(A)}\right]\prod_{i=1}^{N}\psi_i(x_i)+\gamma \qquad \psi_i(\cdot)=\psi_i(\cdot)^{-1};$$

and

$$\frac{1}{2}\, g(y_1, ..., y_N, e_{N+1}) = \prod_{i=1}^{N} \left[\psi_i(y_i) + \psi_i(y_i)^{-1}\right].$$

It follows from (3.18b) that

$$g(Y) = \prod_{i=1}^{N+1} \left[\psi_i(y_i) + \psi_i(y_i)^{-1}\right]. \tag{3.20}$$

Moreover, either $\gamma = 0$ or $g(y_1, ..., y_N, e_{N+1}) \equiv 2^{N+1}$. In the latter case, (3.20) shows that $\psi_i = 1$ $(i = 1, 2, .., N)$. Hence, (3.19a) is impossible and (3.19b) reduces to $A(X) = S^{(A)}(X) + \alpha^{(A)} + \gamma$. Thus (absorbing γ into $\alpha^{(A)}$ if needed) we may drop γ from (3.19) in either case and write

$$\begin{aligned} &A(x_1, ..., x_N) = \\ &= \sum_{\sigma_1,\ldots,\sigma_k=\pm 1} \left[S^{(A)}_{\sigma_1,\ldots,\sigma_k}(x_{p_1}, ..., x_{p_{N-k}}) + \alpha^{(A)}_{\sigma_1,\ldots,\sigma_k}\right] \times \\ &\times \prod_{i=1}^{N-k} \psi_{p_i}(x_{p_i}) \prod_{j=1}^{k} \psi_{r_j}(x_{r_j})^{\sigma_j}, \end{aligned} \tag{3.21a}$$

or

$$A(X) = \left[S^{(A)}(X) + \alpha^{(A)}\right] \prod_{i=1}^{N} \psi_i(x_i), \qquad \psi_i(\cdot) = \psi_i(\cdot)^{-1}. \tag{3.21b}$$

Note that we know that g has the form (3.20) with determined nonzero exponentials $\psi_1, ..., \psi_{N+1}$. Assuming that A has the form (3.21a), it follows from (3.16) and the inductive hypothesis that

$$\begin{aligned} &B(x_1, ..., x_N) = \\ &= \sum_{\sigma_1,\ldots,\sigma_k=\pm 1} \left[S^{(B)}_{\sigma_1,\ldots,\sigma_k}(x_{p_1}, ..., x_{p_{N-k}}) + \alpha^{(B)}_{\sigma_1,\ldots,\sigma_k}\right] \times \\ &\times \prod_{i=1}^{N-k} \psi_{p_i}(x_{p_i}) \prod_{j=1}^{k} \psi_{r_j}(x_{r_j})^{\sigma_j}, \end{aligned} \tag{3.22}$$

for the same $\{p_1, ..., p_{N-k}\}$ and $\{r_1, ..., r_k\}$ as in (3.21a). Recalling that $\psi_{N+1}(x) \not\equiv \psi_{N+1}(x)^{-1}$, we define $r_{k+1} := N + 1$. Then (3.5), (3.21a), (3.22), (3.20) and (3.17) give (3.3a) for $n = N+1$, as soon as we define

$$S_{\sigma_1,\dots,\sigma_k,1} := S^{(A)}_{\sigma_1,\dots,\sigma_k}, \qquad S_{\sigma_1,\dots,\sigma_k,-1} := S^{(B)}_{\sigma_1,\dots,\sigma_k},$$
$$\alpha_{\sigma_1,\dots,\sigma_k,1} := \alpha^{(A)}_{\sigma_1,\dots,\sigma_k}, \qquad \alpha_{\sigma_1,\dots,\sigma_k,-1} := \alpha^{(B)}_{\sigma_1,\dots,\sigma_k}.$$

Similarly, if A is given by (3.21b), we get a solution of the form (3.3b).

Case 3. Suppose finally that f has the form (3.6). Substituting (3.6) into (GRE) and simplifying, using the additivity of A_{N+1} in its last variable and the fact that $\psi_{N+1}(x) \equiv \psi_{N+1}(x)^{-1} \not\equiv 1$, we deduce that

$$\begin{aligned} &2 \sum_{\sigma_1,\dots,\sigma_N=\pm1} [A_{N+1}(x_1 y_1^{\sigma_1},\dots,x_N y_N^{\sigma_N}, x_{N+1}) + \\ &\quad + B(x_1 y_1^{\sigma_1},\dots,x_N y_N^{\sigma_N})]\, \psi_{N+1}(x_{N+1})\, \psi_{N+1}(y_{N+1}) + 2^{N+1}\gamma = \\ &= \Big\{ [A_{N+1}(x_1,\dots,x_{N+1}) + \\ &\quad + B(x_1,\dots,x_N)] \psi_{N+1}(x_{N+1}) + \gamma \Big\}\, g(Y) + h(Y). \end{aligned} \tag{3.23}$$

Again, considerations of linear independence lead to

$$\begin{aligned} &\sum_{\sigma_1,\dots,\sigma_N=\pm1} A_{N+1}(x_1 y_1^{\sigma_1},\dots,x_N y_N^{\sigma_N}, x_{N+1}) = \\ &\quad = A_{N+1}(X)\, g(Y)\, \left(2\psi_{N+1}(y_{N+1})\right)^{-1}. \end{aligned} \tag{3.24}$$

Suppose that $A_{N+1} \neq 0$. Applying the induction hypothesis, we conclude similar to Case 2 that

$$g(Y) = \prod_{i=1}^{N} [\psi_i(y_i) + \psi_i(y_i)^{-1}]\, [2\psi_{N+1}(y_{N+1})] \tag{3.25}$$

and that either

$$\begin{aligned} &A_{N+1}(x_1,\dots,x_{N+1}) = \\ &= \sum_{\sigma_1,\dots,\sigma_k=\pm1} \Big[S^{(A)}_{\sigma_1,\dots,\sigma_k}(x_{p_1},\dots,x_{p_{N-k}}; x_{N+1}) + \\ &\quad + \alpha^{(A)}_{\sigma_1,\dots,\sigma_k}(x_{N+1}) \Big] \prod_{i=1}^{N-k} \psi_{p_i}(x_{p_i}) \prod_{j=1}^{k} \psi_{r_j}(x_{r_j})^{\sigma_j}, \end{aligned} \tag{3.26a}$$

or

$$A_{N+1}(X) = \tag{3.26b}$$

$$= \left[S^{(A)}(x_1, ..., x_N; x_{N+1}) + \alpha^{(A)}(x_{N+1})\right] \prod_{i=1}^{N} \psi_i(x_i), \psi_i(\cdot) = \psi_i(\cdot)^{-1},$$

where each $S^{(A)}_{\sigma_1,...,\sigma_k}$ is a SMAF of order $N-k$ in the first $N-k$ variables and additive in the last variable, $S^{(A)}$ is a SMAF of order N in the first N variables and additive in the last variable, and $\alpha^{(A)}$ and each $\alpha^{(A)}_{\sigma_1,...,\sigma_k}$ are additive.

Since $\psi_{N+1} \neq 1$, then linear independence, (3.23), and (3.25) draw also

$$\sum_{\sigma_1,...,\sigma_N=\pm 1} B(x_1 y_1^{\sigma_1}, ..., x_N y_N^{\sigma_N}) = \tag{3.27}$$

$$= B(x_1, ..., x_N) \prod_{i=1}^{N} [\psi_i(y_i) + \psi_i(y_i)^{-1}],$$

$$h(Y) = \gamma\{2^{N+1} - g(Y)\}. \tag{3.28}$$

Similar to Case 2, (3.27) leads to the form (3.22) for B with the same $\{p_1, ..., p_{N-k}\}$ and $\{r_1, ..., r_k\}$ as in (3.26). From (3.22), (3.26), (3.6), (3.25) and (3.28), we arrive at solution (3.3) again for $n = N+1$, as soon as we define $p_{N-k+1} := N+1$,

$$S_{\sigma_1,...,\sigma_k}(x_{p_1}, ..., x_{p_{N-k+1}}) := S^{(A)}_{\sigma_1,...,\sigma_k}(x_{p_1}, ..., x_{p_{N-k}}; x_{N+1}) +$$

$$+ \alpha^{(A)}_{\sigma_1,...,\sigma_k}(x_{N+1}) + S^{(B)}_{\sigma_1,...,\sigma_k}(x_{p_1}, ..., x_{p_{N-k}}),$$

and $\alpha_{\sigma_1,...,\sigma_k} := \alpha^{(B)}_{\sigma_1,...,\sigma_k}$, for each $\sigma_1, ..., \sigma_k = \pm 1$.

The only other possibility is that $A_{N+1} = 0$. Now (3.6) takes the form

$$f(x_1, ..., x_{N+1}) = B(x_1, ..., x_N)\, \psi_{N+1}(x_{N+1}) + \gamma, \tag{3.29}$$

with $\psi_{N+1}(\cdot) = \psi_{N+1}(\cdot)^{-1} \neq 1$. Here we know that $B \neq 0$, since f is nonconstant. We show that the solution of (GRE) is of the form (3.3).

Consider (3.23) with $A_{N+1} = 0$. Since ψ_{N+1} and 1 are linearly independent, we find that

$$\sum_{\sigma_1,\ldots,\sigma_N=\pm 1} B\left(x_1 y_1^{\sigma_1}, ..., x_N y_N^{\sigma_N}\right) = \\ = B(x_1, ..., x_N)\, g(Y)\, \left(2\,\psi_{N+1}(y_{N+1})\right)^{-1}, \tag{3.30}$$

and that

$$h(Y) = \gamma\left[2^{N+1} - g(Y)\right],$$

the latter of which is in agreement with a solution of type (3.3) for $n = N+1$. Applying the induction hypothesis to (3.30), we deduce that the solution must be among three possible forms.

The first possibility is that B is a nonzero constant α and $g(Y)\,[2\,\psi_{N+1}(y_{N+1})]^{-1} \equiv 2^N$. In this case, (3.29) shows that we have $f(X) = \alpha\,\psi_{N+1}(x_{N+1}) + \gamma$, a solution of type (3.3).

The second possibility is that $B(X) = S(x_1, ..., x_N) + \alpha$ and $g(Y)\,[2\,\psi_{N+1}(y_{N+1})]^{-1} \equiv 2^N$. By (3.29), we again have a solution of the form (3.3).

The third and final possibility is that either

$$B(X) = \sum_{\sigma_1,\ldots,\sigma_k=\pm 1}\left[S^{(B)}_{\sigma_1,\ldots,\sigma_k}\left(x_{p_1}, ..., x_{p_{N-k}}\right) + \alpha^{(B)}_{\sigma_1,\ldots,\sigma_k}\right] \times \\ \times \prod_{i=1}^{N-k}\psi_{p_i}(x_{p_i}) \prod_{j=1}^{k}\psi_{r_j}(x_{r_j})^{\sigma_j}$$

or

$$B(X) = \left[S^{(B)}(x_1, ..., x_N) + \alpha\right]\prod_{i=1}^{N}\psi_i(x_i) \qquad \psi_i(\cdot) = \psi_i(\cdot)^{-1},$$

and

$$g(Y)\ \left[2\,\psi_{N+1}(y_{N+1})\right]^{-1} = \prod_{i=1}^{N}\left[\psi_i(y_i) + \psi_i(y_i)^{-1}\right].$$

Again, (3.29) shows that f and g have the form specified in (3.3) for $n = N+1$.

This exhausts all possible cases. Since the converse is easy to check, this concludes the proof of the theorem.

4. General solution of (GRH)

We solve (GRH) by showing first that any f satisfying (GRH) satisfies also (GRE).

Lemma 2. *If* $f, p, q : \mathbf{G}_1 \times \cdots \times \mathbf{G}_n \to \mathbf{K}$ *satisfy* (GRH), *then* f, g, h *satisfy* (GRE) *with* g, h *given by*

$$g(Y) = \prod_{k=1}^{n} \{p[y_k] - 2\,(n-1)\} \tag{4.1a}$$

$$h(Y) = \sum_{k=1}^{n} 2^{n-k}\, q[y_k] \prod_{i=1}^{k-1} \{p[y_i] - 2\,(n-1)\}\,, \tag{4.1b}$$

where $[y_k]$ *represents* $(e_1, e_2, ..., e_{k-1}, y_k, e_{k+1}, ..., e_n)$.

Proof. First, assuming that f, p, q satisfy (GRH), put $y_i = e_i$ $(i \neq k)$ to get

$$\begin{aligned} &\sum_{\sigma_k = \pm 1} f(x_1, ..., x_{k-1}, x_k y_k^{\sigma_k}, x_{k+1}, ..., x_n) = \\ &= f(x_1, ..., x_n)\, \{p[y_k] - 2\,(n-1)\} + q[y_k], \end{aligned} \tag{4.2}$$

for each fixed $k \in \{1, ..., n\}$.

Next, we establish by induction on j that

$$\begin{aligned} &\sum_{\sigma_1, \dots, \sigma_n = \pm 1} f\left(x_1 y_1^{\sigma_1}, ..., x_n y_n^{\sigma_n}\right) = \\ &= \sum_{\sigma_j, \dots, \sigma_n = \pm 1} f\left(x_1, ..., x_{j-1}, x_j y_j^{\sigma_j}, ..., x_n y_n^{\sigma_n}\right) \times \\ &\times \prod_{i=1}^{j-1} \{p[y_i] - 2\,(n-1)\} + \sum_{k=1}^{j-1} 2^{n-k} q[y_k] \prod_{i=1}^{k-1} \{p[y_i] - 2\,(n-1)\}, \end{aligned} \tag{4.3}$$

for each $j = 1, .., n$. For $j = 1$, (4.3) is an identity. Suppose that (4.3) holds for $j = J \in \{1, ..., n-1\}$. Applying (4.2), we find the right side of (4.3) can be written as

$$\sum_{\sigma_{J+1}, \dots, \sigma_n = \pm 1} \sum_{\sigma_J = \pm 1} f\left(x_1, ..., x_{J-1}, x_J y_J^{\sigma_J}, ..., x_n y_n^{\sigma_n}\right) \prod_{i=1}^{J-1} \{p[y_i] - 2\,(n-1)\}$$

$$+\sum_{k=1}^{J-1} 2^{n-k}\, q[y_k] \prod_{i=1}^{k-1} \{p[y_i] - 2\,(n-1)\}$$

$$= \sum_{\sigma_{J+1},\ldots,\sigma_n=\pm 1} \left\{ f\left(x_1, ..., x_J, x_{J+1} y_{J+1}^{\sigma_{J+1}}, ..., x_n y_n^{\sigma_n}\right)\, \{p[y_J] - 2\,(n-1)\}\right.$$

$$\left. + \, q\, [y_J] \right\} \prod_{i=1}^{J-1} \{p[y_i]{-}2\,(n-1)\} + \sum_{k=1}^{J-1} 2^{n-k} q[y_k] \prod_{i=1}^{k-1} \{p[y_i] - 2\,(n-1)\}$$

$$= \sum_{\sigma_{J+1},\ldots,\sigma_n=\pm 1} f\left(x_1, ..., x_J, x_{J+1} y_{J+1}^{\sigma_{J+1}}, ..., x_n y_n^{\sigma_n}\right) \prod_{i=1}^{J} \{p[y_i] - 2\,(n-1)\}$$

$$+ 2^{n-J} q[y_J] \prod_{i=1}^{J-1} \{p[y_i]{-}2\,(n-1)\} + \sum_{k=1}^{J-1} 2^{n-k} q[y_k] \prod_{i=1}^{k-1} \{p[y_i] - 2\,(n-1)\},$$

which gives (4.3) for $j = J+1$. Thus (4.3) is valid for $j = 1, .., n$.

Finally, observe that (4.3) for $j = n$ is

$$\sum_{\sigma_1,\ldots,\sigma_n=\pm 1} f\left(x_1 y_1^{\sigma_1}, ..., x_n y_n^{\sigma_n}\right) =$$

$$= \sum_{\sigma_n=\pm 1} f\left(x_1, ..., x_{n-1}, x_n y_n^{\sigma_n}\right) \prod_{i=1}^{n-1} \{p[y_i] - 2\,(n-1)\} +$$

$$+ \sum_{k=1}^{n-1} 2^{n-k} q[y_k] \prod_{i=1}^{k-1} \{p[y_i] - 2\,(n-1)\}.$$

Applying (4.2) once more (for $k = n$) on the right hand side of this equation, and defining g, h by (4.1a) and (4.1b), we have (GRE). This finishes the proof.

Now we are ready for the second main result, which is the following.

Theorem 2. *The general solution* $f, p, q : \mathbf{G}_1 \times \cdots \times \mathbf{G}_n \to \mathbf{K}$ *of* (GRH), *with f satisfying the factorization condition* (FC) *in each variable, is given by*

(4.4) $\qquad f(X) = \gamma, \qquad p$ *arbitrary,* $\qquad q(Y) = \gamma[2n - p(Y)],$

(4.5) $\quad f(X) = S(X) + \alpha + \sum_{i=1}^{n} Q_i(x_i), \quad p(Y) = 2n, \quad q(Y) = 2 \sum_{i=1}^{n} Q_i(y_i),$

$$
\text{(4.6a)}\quad \begin{cases} f(X) = \displaystyle\sum_{\sigma_1,\dots,\sigma_k=\pm1} \big[S_{\sigma_1,\dots,\sigma_k}(x_{p_1},\dots,x_{p_{n-k}})+ \\ \qquad\qquad +\alpha_{\sigma_1,\dots,\sigma_k}\big] \displaystyle\prod_{i=1}^{n-k} \psi_{p_i}(x_{p_i}) \prod_{j=1}^{k} \psi_{r_j}(x_{r_j})^{\sigma_j} + \gamma \\ p(Y) = \displaystyle\sum_{i=1}^{n} \big[\psi_i(y_i) + \psi_i(y_i)^{-1}\big], \quad q(Y) = \gamma\,[2n - p(Y)] \end{cases}
$$

$$
\text{(4.6b)}\quad \begin{cases} f(X) = [S(X)+\alpha] \displaystyle\prod_{i=1}^{n} \psi_i(x_i) + \gamma, \\ p(Y) = 2 \displaystyle\sum_{i=1}^{n} \psi_i(y_i) \not\equiv 2n, \\ \qquad q(Y) = \gamma\ [2n - p(Y)], \qquad \psi_i(\cdot) = \psi_i(\cdot)^{-1} \end{cases}
$$

where k is a fixed integer ($1 \le k \le n$), α, γ and $\alpha_{\sigma_1,\dots,\sigma_k}$ ($\sigma_1,\dots,\sigma_k = \pm1$) are arbitrary constants, S and $S_{\sigma_1,\dots,\sigma_k}$ ($\sigma_1,\dots,\sigma_k = \pm1$) are SMAF's *of order n and $n-k$, respectively, Q_i is quadratic and ψ_i is a nonzero exponential ($1 \le i \le n$), ψ_{p_i} ($i = 1,2,\dots,n-k$) satisfies also $\psi_{p_i}(x) \equiv \psi_{p_i}(x)^{-1}$, ψ_{r_j} ($j = 1,2,\dots,k$) satisfies also $\psi_{r_j}(x) \not\equiv \psi_{r_j}(x)^{-1}$, and $\{1,2,\dots,n\}$ is the (disjoint) union of the sets $\{p_1,\dots,p_{n-k}\}$ and $\{r_1,\dots,r_k\}$.*

Here again we have adopted the convention (C1) in (4.6a) (cf. Theorem 1).

Proof. If f satisfies (GRH), then Lemma 2 shows that it also satisfies an equation of the form (GRE). Since f also satisfies (FC), we may apply Theorem 1 to obtain the possible forms of f. We consider the three forms of f case by case.

First, suppose f is constant, as in (3.1). Substituting $f(X) = \gamma$ into (GRH), we get $2n\gamma = \gamma\, g(Y) + h(Y)$, which gives solution (4.4). Henceforth, we assume that f is nonconstant.

Secondly, suppose that f has the form given in (3.2). Inserting this into (GRH) and simplifying, we arrive at

$$
\text{(4.7)}\quad \begin{aligned} &2n\,[S(X) + \alpha + \sum_{i=1}^{n} Q_i(x_i)] + 2\sum_{i=1}^{n} Q_i(y_i) = \\ &\quad = [S(X) + \alpha + \sum_{i=1}^{n} Q_i(x_i)]g(Y) + h(Y), \end{aligned}
$$

where we have used the facts that S is a SMAF of order n and that Q_i $(i = 1, 2, ..., n)$ is quadratic. Since f is nonconstant, at least one of the functions S, $Q_1, ..., Q_n$ must be nonzero. Thus, by a linear independence argument, we obtain from (4.7) that $g(Y) = 2n$ and $h(Y) = 2\sum_{i=1}^{n} Q_i(y_i)$. That is, we have solution (4.5).

Finally, suppose that f is as in (3.3). Substituting form of f from (3.3a) into (GRH) and simplifying, we come to

$$
\begin{aligned}
(4.8) \quad & \sum_{\sigma_1,...,\sigma_k=\pm 1} \big[S_{\sigma_1,...,\sigma_k}(x_{p_1}, ..., x_{p_{n-k}}) + \\
& \quad + \alpha_{\sigma_1,...,\sigma_k}\big] \prod_{i=1}^{n-k} \psi_{p_i}(x_{p_i}) \prod_{j=1}^{k} \psi_{r_j}(x_{r_j})^{\sigma_j} \left\{ \sum_{i=1}^{n-k} 2\, \psi_{p_i}(y_{p_i}) + \right. \\
& \quad \left. + \sum_{j=1}^{k} [\psi_{r_j}(y_{r_j}) + \psi_{r_j}(y_{r_j})^{-1}] \right\} + 2n\gamma \\
& = \left\{ \sum_{\sigma_1,...,\sigma_k=\pm 1} \big[S_{\sigma_1,...,\sigma_k}(x_{p_1}, ..., x_{p_{n-k}}) + \right. \\
& \quad \left. + \alpha_{\sigma_1,...,\sigma_k}\big] \prod_{i=1}^{n-k} \psi_{p_i}(x_{p_i}) \prod_{j=1}^{k} \psi_{r_j}(x_{r_j})^{\sigma_j} + \gamma \right\} g(Y) + h(Y).
\end{aligned}
$$

Here we have used the facts that $S_{\sigma_1,...,\sigma_k}$ $(\sigma_1, ..., \sigma_k = \pm 1;\ 1 \leq k \leq n)$ is a SMAF of order $n - k$, that $\psi_{p_i}(x) \equiv \psi_{p_i}(x)^{-1}$ $(i = 1, ..., n-k)$, and that each ψ_i is nonzero exponential $(i = 1, ..., n)$. Again using linear independence considerations, we conclude from (4.8) that

$$
g(Y) = \sum_{i=1}^{n-k} 2\, \psi_{p_i}(y_{p_i}) + \sum_{j=1}^{k} [\psi_{r_j}(y_{r_j}) + \psi_{r_j}(y_{r_j})^{-1}]
$$

and that $h(Y) = 2n\gamma - \gamma g(Y)$.

Thus we have (4.6a). Similarly, if f has the form in (3.3b), then we arrive at (4.6b). This completes the proof.

5. Some further results

We complete the link between (GRE) and (GRH) with the following converse to Lemma 2.

Lemma 3. *If* $f, g, h : \mathbf{G}_1 \times \mathbf{G}_2 \times \cdots \times \mathbf{G}_n \to \mathbf{K}$ *satisfy* (GRE) *then* f, p, q *satisfy* (GRH) *with* p, q *given by*

$$p(Y) = 2^{1-n} \sum_{i=1}^{n} g[y_i], \tag{5.1a}$$

$$q(Y) = 2^{1-n} \sum_{i=1}^{n} h[y_i], \tag{5.1b}$$

where, as before, $[y_i] = (e_1, ..., e_{i-1}, y_i, e_{i+1}, ..., e_n)$.

Proof. For each fixed $i \in \{1, ..., n\}$, setting $y_j = e_j$ for all $j \neq i$ in (GRE), we get

$$2^{n-1} \sum_{\sigma_i = \pm 1} f(x_1, ..., x_{i-1}, x_i y_i^{\sigma_i}, x_{i+1}, ..., x_n) = f(X)\, g[y_i] + h[y_i].$$

Summing over i, we have

$$2^{n-1} \sum_{i=1}^{n} \sum_{\sigma_i = \pm 1} f(x_1, ..., x_{i-1}, x_i y_i^{\sigma_i}, x_{i+1}, ..., x_n) =$$

$$= f(X) \sum_{i=1}^{n} g[y_i] + \sum_{i=1}^{n} h[y_i],$$

which is (GRH) with p, q given by (5.1a), (5.1b).

This lemma provides a new way of proving Theorem 2. Namely, for nonconstant f, we apply the formulas (5.1a), (5.1b) to (3.2)–(3.3), obtaining thereby (4.5)–(4.6), respectively.

It is a straightforward job to work out the continuous solutions of (GRE) and (GRH) on $\mathfrak{R}^n$, based on Theorems 1 and 2. Such continuous quadratic, additive, and nonzero exponential functions from $\mathfrak{R}$ into $\mathbb{C}$ (the field of complex numbers) are of the forms

$$Q(x) = bx^2, \qquad A(x) = ax, \qquad \psi(x) = e^{cx},$$

respectively, for complex constants a, b, c. We call any function $P : \mathfrak{R}^n \to \mathbb{C}$ of the form

$$P(x_1, ..., x_n) = a_{12\cdots n} \prod_{i=1}^{n} x_i + a_{12\cdots n-1} \prod_{i=1}^{n-1} x_i + \cdots +$$

$$+ a_{23\cdots n} \prod_{i=2}^{n} x_i + \cdots + a_n x_n + a_0$$

a *sum of multilinear functions* (SMULF) of degree n. It can be shown [4] that if $S : \mathfrak{R}^n \to \mathbb{C}$ is a regular SMAF of order n, then for any constant α, $S + \alpha$ is a SMULF of degree n. Moreover, it is also true that f is continuous in (3.2) and (3.3), (4.5) and (4.6) if and only if each SMAF, each quadratic Q_i, and each nonzero exponential ψ_i is continuous.

Therefore, one obtains the continuous versions of Theorems 1 and 2 on $\mathfrak{R}^n$ by replacing each SMAF by an appropriate SMULF, each $Q_i(x)$ by $b_i x^2$, each ψ_{p_i} by 1, and each $\psi_{r_j}(x)$ by $e^{c_{r_j} x}$ $(c_{r_j} \neq 0)$.

It is also possible (but rather tedious) to work out explicit forms of the real-valued solutions of (GRE) and (GRH) on $\mathfrak{R}^n$. Since $\mathfrak{R}$ is not quadratically closed, Theorems 1 and 2 do not immediately apply in this case. One may proceed by screening the complex-valued solutions to obtain those which are real-valued. As an illustration, we describe the process for finding the continuous real-valued solutions of (GRE) for $n = 2$. It turns out that the Q_i's and the SMULF's have the same form as above, but with real constants. From (3.1) and (3.2), we obtain respectively

$$\begin{cases} f(x_1, x_2) = \gamma, \\ \qquad g \quad \text{arbitrary} \quad \text{real-valued} \quad \text{function} \\ h(y_1, y_2) = \gamma\,[4 - g(Y)]\,; \end{cases}$$

$$\begin{cases} f(x_1, x_2) = a_{12}x_1x_2 + a_1x_1 + a_2x_2 + a_0 + b_1x_1^2 + b_2x_2^2, \\ g(y_1, y_2) = 4, \\ h(y_1, y_2) = 4\left(b_1x_1^2 + b_2x_2^2\right), \end{cases}$$

where all constants are real. Case (3.3b) is impossible, since each $\psi_i = 1$ but $g(Y) \not\equiv 2^n$. The situation for (3.3a) is more complicated. Each ψ_{p_i} is equal to 1, but there are two alternatives for each ψ_{r_j}. Because g must be real-valued, when $\psi(x) \not\equiv \psi(x)^{-1}$, we have either $\psi(x) = e^{bx}$ for real $b\,(\neq 0)$ or $\psi(x) = \cos bx + \mathbf{i} \sin bx$ for some real $b\,(\neq 0)$, where $\mathbf{i}$ is the imaginary unit. When $k = 1$, the solutions are given by

$$\begin{cases} f(x_1, x_2) = (a_1 x_2 + \alpha_1)\, e^{cx_1} + (a_2 x_2 + \alpha_2)\, e^{-cx_1} + \gamma, \\ g(y_1, y_2) = 2\left(e^{cy_1} + e^{-cy_1}\right), \\ h(y_1, y_2) = \gamma\,[4 - g(Y)]\,; \end{cases}$$

$$\begin{cases} f(x_1, x_2) = (a_3 x_2 + \alpha_3) \cos bx_1 + (a_4 x_2 + \alpha_4) \sin bx_1 + \gamma, \\ g(y_1, y_2) = 4 \cos by_1, \\ h(y_1, y_2) = \gamma\,[4 - g(Y)]\,, \end{cases}$$

where all constants are real, and two similar forms obtained by interchanging x_1 with x_2 and y_1 with y_2. Finally, when $k = 2$ in (3.3a) there are again four forms, since each of ψ_1 and ψ_2 can have either of the forms e^{bx} or e^{ibx} for some real $b\,(\neq 0)$.

We conclude the paper with an example illustrating the non-equivalence of (GRE) and (GRH) when char $\mathbf{K} = 2$.

Example. Let $\mathbf{G}_1 = (\mathbf{Z}_2, +)$, $\mathbf{G}_2 = (\Re, +)$, $\mathbf{K} = \mathbf{Z}_2$, and define $f : \mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{K}$ by

$$f(x_1, x_2) = \begin{cases} 1 & \text{if } x_2 \neq 0 \\ 0 & \text{if } x_2 = 0. \end{cases}$$

Then the left side of (GRE) is

$$\begin{aligned} f(x_1 + y_1, x_2 + y_2) &+ f(x_1 - y_1, x_2 + y_2) + \\ &+ f(x_1 + y_1, x_2 - y_2) + f(x_1 - y_1, x_2 - y_2) \\ &= 2\, f(x_1 + y_1, x_2 + y_2) + 2\, f(x_1 + y_1, x_2 - y_2) = 0, \end{aligned}$$

so f satisfies (GRE) with $g = h = 0$. But f does not satisfy (GRH) at all. Indeed, suppose that f did satisfy (GRH) with some p, q. Then, as $f(x_1 + y_1, x_2) + f(x_1 - y_1, x_2) = 2\, f(x_1 + y_1, x_2) = 0$, (GRH) would become

$$f(x_1, x_2 + y_2) + f(x_1, x_2 - y_2) = f(x_1, x_2)\, p(y_1, y_2) + q(y_1, y_2). \tag{5.2}$$

Putting $x_2 = 0$ in (5.2), we obtain $q(y_1, y_2) = 0$. Using this with $y_2 = 1$ in (5.2), we get

$$f(x_1, x_2 + 1) + f(x_1, x_2 - 1) = f(x_1, x_2)\, p(y_1, 1).$$

But this equation yields, with $x_2 = 1, 2$, respectively,

$$p(y_1, 1) = 1 \qquad \text{and} \qquad p(y_1, 1) = 0,$$

a contradiction.

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