On systems of polynomials orthogonal in two intervals.

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§ 1.

Let us take the so-called HEUN's differential equation:

$$(1)\qquad x(x-y)(x-z)\left[\frac{d^2u}{dx^2}+\left(\frac{A}{x}+\frac{B}{x-y}+\frac{C}{x-z}\right)\frac{du}{dx}\right]+(\varkappa x+\lambda)u=0$$

where $A, B, C, \varkappa, \lambda, y, z$ are constants and look for the conditions securing a polynomial solution.

If there exists a solution of the form

$$u=x^n+\alpha x^{n-1}+\beta x^{n-2}+\ldots+\eta x+\vartheta$$

and if we substitute this polynomial into equation (1) then there will stand on the left hand side an expression of degree $n+1$ the highest term of which is $[n(n-1)+(A+B+C)n+\varkappa]x^{n+1}$.

The necessary condition for the existence of a polynomial solution of the degree n is therefore

$$(2)\qquad \varkappa=\varkappa_n=-n(n-1)-(A+B+C)n.$$

If A, B and C are non-negative numbers further e. g. $y<0<z$ and condition (2) is fulfilled then a theorem of HEINE and STIELTJES[1]) asserts that there are exactly $n+1$ distinct values of λ such that equation (1) should have a polynomial solution of degree n. If these polynomials are denoted by

$$P_{n,0}(x),\ P_{n,1}(x),\ldots,P_{n,n}(x)$$

then according to the same theorem the indices of these polynomfals can be chosen in such a way that *there be exactly k roots of the polynomial* $P_{n,k}(x)$ *in the interval* $(y,0)$ *and exactly* $n-k$ *roots in the interval* $(0,z)$.

Let now be

$$T=T(x)=x(x-y)(x-z),\ \varrho=\varrho(x)=x^{A-1}(x-y)^{B-1}(x-z)^{C-1}$$
$$(A>0, B>0, C>0).$$

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1) SZEGŐ, G.: Orthogonal Polynomials. (New York, 1939.) p. 147. — STIELTJES: Sur certains polynômes ... et sur la théorie des fonctions de Lamé. *Acta Math.* 6 (1885), pp. 321-326.

The polynomials $\{P_{n,k}(x)\}_{k=0}^{k=n}$ have an interesting property: *they are orthogonal in either of the two intervals* $(y,0)$ *and* $(0,z)$ *with the weight function* $\varrho(x)$:

$$(3) \qquad \int_y^0 P_{n,i}(x)P_{n,k}(x)\varrho\, dx = \int_0^z P_{n,i}(x)P_{n,k}(x)\varrho\, dx = 0 \quad (i \neq k).$$

For proving this let us multiply equation (1) with $\varrho(x)$. Then it is to be seen that $P_{n,i} = P_{n,i}(x)$ and $P_{n,k} = P_{n,k}(x)$ satisfy the equations

$$(4) \qquad (T\varrho P'_{n,i})' + (\varkappa_n x + \lambda_{n,i})\varrho P_{n,i} = 0$$

and

$$(5) \qquad (T\varrho P'_{n,k})' + (\varkappa_n x + \lambda_{n,k})\varrho P_{n,k} = 0,$$

where $\lambda_{n,i} \neq \lambda_{n,k}$. From that the relations of orthogonality (3) follow in the usual way.

§ 2.

Let us now consider the following more general question. Given two real functions $\varrho_1(x)$ and $\varrho_2(x)$ not changing sign in the domains of integration I_1 resp. I_2: what are the conditions to be imposed on $\varrho_1(x)$ and $\varrho_2(x)$ if we want that there should exist a set of polynomials

$$P_0(x),\ P_1(x), \dots, P_n(x)$$

of degree at most n satisfying the conditions of orthogonality in *two* domains

$$(6) \qquad \int_{I_1} P_i(x)P_k(x)\varrho_1(x)\, dx = \int_{I_2} P_i(x)P_k(x)\varrho_2(x)\, dx = 0 \quad (i \neq k)?$$

It will be proved that *the necessary and sufficient condition for the existence of the set* $\{P_k(x)\}_{k=0}^{k=n}$ *is that the moments*

$$(7) \qquad M_\nu^{(1)} = \int_{I_1} x^\nu \varrho_1(x)\, dx \text{ and } M_\nu^{(2)} = \int_{I_2} x^\nu \varrho_2(x)\, dx \ (\nu = 0, 1, \dots, 2\,n)$$

exist the integral being taken in LEBESGUE's sense. If conditions (7) are fulfilled then to a given pair $\varrho_1 = \varrho_1(x)$, $\varrho_2 = \varrho_2(x)$ of weight functions there exists generally one and only one set $\{P_k(x)\}_{k=0}^{k=n}$ if the elements of the set are normalized in any suitable way.

Moreover the way of the proof makes it clear that this result cannot be generalized to more than two intervals resp. weight functions.

The sufficiency of the condition. Let $\{p_k(x)\}_{k=0}^{k=n}$ be an orthonormal set of polynomials of degree not higher than n which satisfies the conditions

$$\int_{I_1} p_i(x)p_k(x)\varrho_1\, dx = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}.$$

This set exists provided the moments (7) exist. Consider the quadratic forms

$$(9_1) \qquad Q_1(u,u) = \int_{I_1} \left(\sum_{i=0}^n p_i(x)u_i \right)^2 \varrho_1\, dx$$

and

$$(9_2) \qquad Q_2(u,u)=\int\limits_{I_2}\left(\sum_{i=0}^{n} p_i(x)u_i\right)^2 \varrho_2 dx$$

which exist as a consequence of the assumption. The first of these is equal to $\sum_{i=0}^{n} u_i^2$ while the other contains also terms of mixed type. Now there exists a real linear orthogonal transformation which reduces the quadratic form (9_2) to its principal axes. Let this transformation be

$$u_i=\sum_{k=0}^{n} c_{i,k} v_k \qquad (i=0,1,\ldots,n)$$

and hence

$$Q_2(u,u)=Q_2^*(v,v)=\int\limits_{I_2}\left[\sum_{i=0}^{n} p_i(x)\sum_{k=0}^{n} c_{i,k}v_k\right]^2 \varrho_2 dx=$$

$$=\int\limits_{I_2}\left[\sum_{k=0}^{n}\left(\sum_{i=0}^{n} c_{i,k}p_i(x)\,v_k\right)\right]^2 \varrho_2 dx=\sum_{k=0}^{n}\lambda_k v_k^2.$$

The same transformation leaves the form $Q_1(u,u)$ unaltered:

$$Q_1(u,u)=Q_1^*(v,v)=\int\limits_{I_1}\left[\sum_{k=0}^{n}\left(\sum_{i=0}^{n} c_{i,k}p_i(x)\right)v_k\right]^2 \varrho_1 dx=\sum_{k=0}^{n} v_k^2.$$

Now let $P_k(x)$ be defined by

$$P_k(x)=\sum_{i=0}^{n} c_{i,k}p_i(x) \qquad (k=0,1,\ldots,n).$$

The set $\{P_k(x)\}_{k=0}^{k=n}$ consists of polynomials of degree not exceeding n which satisfy conditions (6) and moreover

$$(10) \qquad \int\limits_{I_1} P_k^2(x)\varrho_1 dx=1, \qquad \int\limits_{I_2} P_k^2(x)\varrho_2 dx=\lambda_k\neq 0.$$

If $\lambda_i\neq\lambda_k$ $(i\neq k)$ then there exists only one transformation which reduces the quadratic form (9_2) to the principal axes apart from trivial permutations of the indices. In this case the set $\{P_k(x)\}_{k=0}^{k=n}$ is defined by the conditions (6) in an unambiguous way.

The necessity of the condition. From the existence of (6) and (10) follows the existence of the moments (7). For the orthogonality of the set $\{P_k(x)\}$ involves the linear independence of its elements. Thus x^μ can be written in the form

$$x^\mu=\sum_{k=0}^{n} a_{\mu,k}P_k(x) \qquad (\mu=0,1,\ldots,n).$$

If ν is a non-negative integer not exceeding $2n$ then x^ν can be written in the form $x^p x^q$ where p and q are integers not exceeding n. At last

$$M_\nu^{(1)} = \int_{I_1} x^p x^q \varrho_1 \, dx = \int_{I_1} \Big(\sum_k a_{p,k} P_k(x)\Big)\Big(\sum_l a_{q,l} P_l(x)\Big) \varrho_1 \, dx$$

where the last integral has evidently a meaning. The existence of the moments $M_\nu^{(2)}$ can be shown in a similar manner.

§ 3.

Let us consider the quantities λ_k in the formula (10). Their value is equal to

$$\int_{I_2} [P_k(x)]^2 \varrho_2 \, dx \Big/ \int_{I_1} [P_k(x)]^2 \varrho_1 \, dx$$

also in that case, when the set $\{P_k(x)\}$ is not normed but satisfies only conditions (6). If $Q = Q(x)$ is any polynomial of degree not exceeding n then

$$\int_{I_2} P_k Q \varrho_2 \, dx = \lambda_k \int_{I_1} P_k Q \varrho_1 \, dx. \tag{11}$$

Indeed, the polynomial $Q(x)$ can be written in the form $Q = \sum c_k P_k$ whence (11) follows immediately.

On the other hand, supposing $\lambda_i \neq \lambda_k$ (11) characterizes the polynomials $P_k(x)$. More exactly if $R(x)$ and $Q(x)$ are polynomials of degrees not exceeding n, further if we vary $Q(x)$ with $R(x)$ remaining fixed then the value of

$$\int_{I_2} R(x) Q(x) \varrho_2 \, dx \Big/ \int_{I_1} R(x) Q(x) \varrho_1 \, dx \tag{12}$$

remains constant if and only if $R(x) = \text{const.}\, P_k(x)$ $(k = 0, 1, \ldots, n)$. For let $R(x) = \sum c_k P_k(x)$ where e. g. $c_0 \neq 0$ and $c_1 \neq 0$. Then the value of the quotient (12) is λ_0 in the case $Q(x) = P_0(x)$ and λ_1 in the case $Q(x) = P_1(x)$.

Another consequence of (11) is that if $Q(x)$ and $P_k(x)$ are orthogonal in I_1 with respect to $\varrho_1(x)$ and $Q(x)$ is a polynomial of degree $\leqq n$ then $Q(x)$ and $P_k(x)$ are orthogonal in the domain I_2 too [with respect to $\varrho_2(x)$].

§ 4.

If the domains I_r are the intervals $a_r \leqq x \leqq b_r$ $(r = 1, 2)$ further for sake of simplicity

$$a_1 < b_1 \leqq 0 \leqq a_2 < b_2$$

then the polynomials of the set $\{P_k(x)\}_{k=0}^{k=n}$ defined by (6) *are exactly of degree n, their roots being real, simple and contained either in the open interval $\langle a_1, b_1 \rangle$ or in the open interval $\langle a_2, b_2 \rangle$.*

Supposing the contrary of this statement: then $P_k(x)$ vanishes in the interior of the above mentioned two intervals at $\nu < n$ different points. Let these points be $x_1, x_2, \ldots, x_\nu$ and the corresponding multiplicities of the roots

$m_1, m_2, \ldots, m_\nu$. Let us consider in the case $\nu = 0$ the function $S(x) \equiv 1$ and in the case $\nu > 0$ the function

$$S(x) = (x - x_1)^{\varkappa_1} \ldots (x - x_\nu)^{\varkappa_\nu}$$

where $\varkappa_l$ is 0 or 1 according as m_l is even or odd.

The degree of the polynomial $S(x)$ is less than n and as the polynomial $S(x)P_k(x)$ doesn't change its sign in either of the intervals (a_1, b_1), (a_2, b_2) the value of the integrals

$$\int_{a_1}^{b_1} S(x)P_k(x)\varrho_1\,dx \quad \text{and} \quad \int_{a_2}^{b_2} S(x)P_k(x)\varrho_2\,dx$$

will be different from 0. On the other hand the polynomial $(x-c)\cdot S(x)$ of a degree not exceeding n and $P_k(x)$ are orthogonal in the interval (a_1, b_1) at a fixed value of c:

$$\int_{a_1}^{b_1} (x-c)\,S(x)\,P_k(x)\varrho_1\,dx = 0$$

whence

$$c = \frac{\int_{a_1}^{b_1} x\,S(x)\,P_k(x)\varrho_1\,dx}{\int_{a_1}^{b_1} S(x)\,P_k(x)\,\varrho_1\,dx}.$$

The right hand side of this equation is the integral mean of x associated with the weight function $|S(x)\,P_k(x)\varrho_1|$ as the integrand of the denominator does not change its sign in the domain of integration. From this

$$a_1 < c < b_1.$$

As according to § 3 the polynomials $(x-c)\,S(x)$ and $P_k(x)$ are orthogonal with respect to ϱ_2 in the interval (a_2, b_2) we can show in the same way that

$$a_2 < c < b_2$$

which is apparently a contradiction.

§ 5.

At last we will show that in the case of the set $\{P_k(x)\}$ the theorem of Heine and Stieltjes is valid in the form proposed in § 1.

If I_1 and I_2 are the intervals specified in the first part of § 4 then the indices of the set $\{P_k(x)\}_{k=0}^{k=n}$ can be chosen in such a way that there should be k roots of $P_k(x)$ in I_1 and $(n-k)$ roots of the same polynomial in I_2.

For proving this let us define the function $w(x)$ in the following way:

$$w(x) = \begin{cases} \varrho_1(x) & \text{if} \quad a_1 \leqq x \leqq b_1 \\ \varrho_2(x) & \text{if} \quad a_2 \leqq x \leqq b_2 \\ 0 & \text{otherwise.} \end{cases}$$

Let us still consider the functions

$$s(x)=\begin{cases}1 & \text{if } a_1 \leqq x \leqq b_2\\ 0 & \text{otherwise}\end{cases}$$

and

$$W(x,t)=(1-t)\,s(x)+t\,w(x).$$

The moments

$$M_\nu^{(r)}(t)=\int_{a_r}^{b_r} x^\nu W(x,t)\,dx \quad (r=1,2;\ \nu=0,1,\ldots,2\,n) \tag{13}$$

exist provided the set $\{M_\nu^{(r)}(1)\}$ exists which is just the necessary condition for the existence of the system $\{P_k(x)\}$. If this condition is fulfilled then to each value of t there exists in the general case a set $\{P_k x,t)\}_{k=0}^{k=n}$ of polynomials which is uniquely defined apart from a constant factor and which satisfies the conditions

$$\int_{a_1}^{0} P_i(x,t)\cdot P_k(x,t)\,W(x,t)\,dx=\int_{0}^{b_1} P_i(x,t)\cdot P_k(x,t)\,W(x,t)\,dx \quad (i\neq k).$$

In the case $t=1$ this set is identical with the set mentioned at the beginning of § 2.

Let now be

$$P_k(x,t)=a_{k,0}(t)+a_{k,1}(t)x+\ldots+a_{k,n-1}(t)x^{n-1}+x^n.$$

The coefficients $a_{k,l}(t)$ can be determined with the aid of formula (11) if we substitute there instead of $Q(x)$ the functions $1, x, x^2, \ldots, x^n$. With regard to (13) we derive from (11) the system of $n+1$ linear equations $[a_{k,n}(t)=1]$

$$\sum_{l=0}^{n}\left[M_{i+l}^{(2)}(t)-\lambda_k M_{i+l}^{(1)}(t)\right]a_{k,l}(t)=0 \qquad (i=0,1,\ldots,n) \tag{14}$$

which has a nontrivial solution if its determinant vanishes. It is to be seen that for each value of k the same determinantal equation of degree $n+1$ serves to the determination of λ_k. As the roots of the determinantal equation are continuous functions of the elements of the determinant, further the elements of the determinant are continuous functions of t we can assert that $\lambda_k=\lambda_k(t)$ is a continuous function of the variable t. But then with regard to the system (14) the quantities $a_{k,l}(t)$ are continuous functions[2]) of t and

[2]) If the rank of the matrix of the system (14) is less than n which can occur only at some isolated values of t, we conclude as follows. Let $t=\tau$ be such an isolated value where the system (14) has more than one linearly independent solution. We then define

$$a_{k,l}(\tau)=\lim_{t\to\tau-0} a_{k,l}(t).$$

As it was seen that at every value of t $a_{k,l}(t)$ is a continuous function of t, therefore in the case $0<\tau<1$ and $\tau<t<\tau+\varepsilon$ where ε is sufficiently small we can define $a_{k,l}(t)$ so as

$$\lim_{t\to\tau+0} a_{k,l}(t)=a_{k,l}(\tau).$$

at last the quantities

$$x_{k,1}(t), \ldots, x_{k,n}(t)$$

which mean the roots of $P_k(x,t)$ are continuous functions of t. In other words: at each value of t the indices of the set $\{x_{k,\nu}(t)\}$ can be chosen in such a way that $x_{k,\nu}(t)$ remains a continuous function of t.

One can show easily that $x_{k,\nu}(t)$ is a bounded function of t. For if e. g. $a_1 < x_{k,\nu}(0) < 0$ then as a consequence of § 4 at each value of t one has $a_1 < x_{k,\nu}(t) < 0$.

If $t=0$, the set $\{P_k(x,t)\}$ is a special case of the STIELTJES set of polynomials mentioned in § 1. ($A=B=C=1$, $a_1=y$, $b_2=z$.) The indices of the set $\{P_k(x,0)\}$ therefore can be chosen in such a way that $P_k(x,0)$ have exactly k roots in $\langle a_1, 0\rangle$ and $n-k$ roots in $\langle 0, b_2\rangle$. But this means by virtue of the aforesaid that if say $x_{k,\nu}(0)$ is within the bounds a_1 and 0 then for each value of t less than 1: $a_1 < x_{k,\nu}(t) < 0$ and consequently $x_{k,\nu}(1)$ is enclosed by the narrower bounds a_1 and b_1. Thus the distribution of the zeros does not change as t increases from 0 to 1: $P_k(x,1)$ has exactly k roots in the interval $\langle a_1, b_1\rangle$ and obviously $n-k$ roots in the interval $\langle a_2, b_2\rangle$.

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