Homologies in a normal space and closed subspace.

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§ 1. Introduction.

Let A be a closed subspace of a normal space R. There are natural homomorphisms of the homology groups of A into those of R. Let $\mathfrak{L}$ denote the kernel of one of these homomorphisms. This article defines and studies groups related to $\mathfrak{L}$. These groups have been studied in [2][1]) when A is a subcomplex of a complex R. The results of [2] have found applications in [3] and [4]. In the present article the results of [2] are extended. Then it is possible to generalize these extended results in the direction of the CECH homology groups and ALEXANDROFF's inner Betti groups.[2])

§ 2. Simultaneous invariants of a complex and subcomplex.

Let K be a space with subspaces L and C. The subspace C will be associated with the special elements of ALEXANDROFF's theory. When the CECH theory is considered, C will be empty. By a simplicial division of K we mean, as in [5], the space K together with a homeomorphism between K and the geometric realization of some finite simplicial complex in a Euclidean space. A simplicial division K^α of K is said to be permissible if the following three conditions hold.

(1) The sets L and C carry subcomplexes L^α and C^α respectively of K^α.

(2) A simplex of K^α is in L^α if all its vertices are in L^α.

(3) If a simplex of K^α has one face in L^α and the opposite face in C^α but not in L^α, the simplex is in C^α.[3])

Let K_1^α, L_1^α, and C_1^α be the first barycentric subdivisions of K^α, L^α, and C^α respectively. Let N_1^α be the complex consisting of the simplexes of K_1^α that have at least one vertex in L_1^α together with the faces of such simplexes.

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[1]) The numbers in brackets refer to the references listed at the end of this article.

[2]) ALEXANDROFF's inner BETTI groups are defined in [1].

[3]) It is easily seen that if a simplicial division D satisfies (1), the first barycentric subdivision of D is permissible. This fact is not used in this article.

Let R_1^α be the complex consisting of the simplexes of K_1^α that have no vertex in L_1^α. Let B_1^α denote the intersection of N_1^α and R_1^α.

Throughout this article it is understood that all chains have as coefficient group a fixed discrete Abelian group. Also the dimension of all cycles and homology classes is fixed at an arbitrary non-negative integer.

Let $\mathfrak{B}^\alpha$ be the subgroup of the homology group of B_1^α mod $B_1^\alpha \cap C_1^\alpha$ made up of the homology classes whose cycles bound in R_1^α mod $R_1^\alpha \cap C_1^\alpha$. Let $\mathfrak{L}^\alpha$ be the subgroup of the homology group of L_1^α mod $L_1^\alpha \cap C_1^\alpha$ made up of the homology classes whose cycles bound in K_1^α mod $K_1^\alpha \cap C_1^\alpha$. Let $\mathfrak{G}^\alpha$ be the subgroup of the homology group of B_1^α mod $B_1^\alpha \cap C_1^\alpha$ made up of the homology classes whose cycles bound both in N_1^α mod $N_1^\alpha \cap C_1^\alpha$ and in R_1^α mod $R_1^\alpha \cap C_1^\alpha$.

Theorem 1. *The groups $\mathfrak{B}^\alpha$, $\mathfrak{L}^\alpha$, and $\mathfrak{G}^\alpha$ are invariant under change of permissible division of K.*

This theorem is proved in [2] for the case that C is empty. Because of condition (3) the proof in [2] generalizes to cover the case that C is not empty. To achieve the generalization one needs only to observe that in all deformations involved in the proof, a point of C never leaves C. Theorem 1 will not be used, and its proof is not given here.

Let N^α, R^α, and B^α be defined in K^α in the same way that N_1^α, R_1^α, and B_1^α are defined in the barycentric subdivision of K^α. Because of (2) any simplex in N^α but not in L^α is the join of a simplex in L^α and a simplex in B^α.[4]) Hence this simplex is made up of segments with end points in L^α and B^α. These segments are called the rays of the simplex. The rays of all such simplexes are called the rays of N^α. Each ray intersects B_1^α in exactly one point, and B_1^α can be homotopically deformed along the rays in N_1^α into L_1^α. Condition (3) implies that a ray intersects $C^\alpha - L^\alpha$ only if the ray lies completely within C^α. Hence during the homotopic deformation just described any point of C^α remains within C^α.

Theorem 2. *We have the isomorphism*

$$\mathfrak{B}^\alpha / \mathfrak{G}^\alpha \simeq \mathfrak{L}^\alpha.$$

Proof. Let $b \in b' \in \mathfrak{B}^\alpha$. Regarding b as a continuous cycle we deform b along the rays into the continuous cycle $\varphi^\alpha b$ in L_1^α. Since during the deformation a point of C^α does not leave C^α, we know that $\varphi^\alpha b$ is a cycle of L_1^α mod $L^\alpha \cap C_1^\alpha$ and that $b \sim \varphi^\alpha b$ in N_1^α mod $N_1^\alpha \cap C_1^\alpha$. It is seen that $\varphi^\alpha b$ bounds in K^α mod C^α. Furthermore $b \sim 0$ in B_1^α mod $B_1^\alpha \cap C_1^\alpha$ implies that $\varphi^\alpha b \sim 0$ in N_1^α mod $N_1^\alpha \cap C_1^\alpha$. This implies that $\varphi^\alpha b \sim 0$ in L_1^α mod $L_1^\alpha \cap C_1^\alpha$ because of the properties of the rays. Thus φ^α determines a homomorphism Φ^α of $\mathfrak{B}^\alpha$ into $\mathfrak{L}^\alpha$.

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[4]) The statements made without proof in the present paragraph are proved in [2].

We show next that Φ^α maps $\mathfrak{B}^\alpha$ upon $\mathfrak{L}^\alpha$. Consider $l \in l' \in \mathfrak{L}^\alpha$ with l simplicial. Let F denote the boundary operator. There is a simplicial chain f of K_1^α such that $Ff = l + c$, c a chain of C_1^α. The chain f is expressible as a sum $f_1 + f_2$ with f_1 a chain of N_1^α and f_2 a chain of R_1^α. Consider Ff_2. Let $Ff_2|B_1^\alpha$ be the chain of B_1^α that has the same value as Ff_2 at each simplex of B_1^α. Since Ff_2 is a chain of $B_1^\alpha \cup C_1^\alpha$, we see that $Ff_2|B_1^\alpha$ is a cycle mod $B_1^\alpha \cap C_1^\alpha$ which bounds in R_1^α mod $R_1^\alpha \cap C_1^\alpha$. This means that $Ff_2|B_1^\alpha$ is in some element of $\mathfrak{B}^\alpha$.

Since f_1 is in N_1^α, we see that $Ff_1 = F(f - f_2) = l + c - Ff_2$ is in N_1^α. But this means that $l - (Ff_2|B_1^\alpha) \sim 0$ in N_1^α mod $N_1^\alpha \cap C_1^\alpha$. Hence $l \sim \varphi^\alpha(Ff_2|B_1^\alpha)$ in N_1^α mod $N_1^\alpha \cap C_1^\alpha$. But the properties of the rays imply that this last homology holds in L_1^α mod $L_1^\alpha \cap C_1^\alpha$. This proves that Φ^α maps $\mathfrak{B}^\alpha$ upon $\mathfrak{L}^\alpha$.

Using again the properties of the rays we easily see that $\varphi^\alpha b \sim 0$ in L_1^α mod $L_1^\alpha \cap C_1^\alpha$ if and only if $b \sim 0$ in N_1^α mod $N_1^\alpha \cap C_1^\alpha$. This fact proves that the kernel of Φ^α is $\mathfrak{G}^\alpha$. Theorem 2 is proved.

§ 3. Permissible mappings.

Let K^β, L^β, and C^β satisfy the conditions (1), (2), and (3) imposed on the complexes with index α. Let N^β, B^β, and R^β be defined for K^β as N^α, etc., are defined for K^α. A simplicial mapping S of K^β into K^α is said to be permissible if the following inclusions hold.

(4) $SC^\beta \subset C^\alpha$, $SL^\beta \subset L^\alpha$, $SN^\beta \subset N^\alpha$, $SB^\beta \subset B^\alpha$, $SR^\beta \subset R^\alpha$. The simplicial mapping S determines a natural mapping S' of a geometric realization of K^β into a geometric realization of K^α. From $SB^\beta \subset B^\alpha$ and $SL^\beta \subset L^\alpha$ it follows that any ray of N^β is mapped by S' upon a ray of N^α.

From S we obtain a simplicial mapping S_1 of K_1^β into K_1^α by mapping the barycenter of a simplex of K^β upon the barycenter of the transform of this simplex by S. It is seen that $S_1 C_1^\beta \subset C_1^\alpha$, $S_1 L_1^\beta \subset L_1^\alpha$, $S_1 N_1^\beta \subset N_1^\alpha$, $S_1 B_1^\beta \subset B_1^\alpha$, and $S_1 R_1^\beta \subset R_1^\alpha$. These inclusions imply the existence of homomorphisms

(5) $$\beta_\alpha^\beta \mathfrak{B}^\beta \subset \mathfrak{B}^\alpha,$$

(6) $$\lambda_\alpha^\beta \mathfrak{L}^\beta \subset \mathfrak{L}^\alpha,$$

(7) $$\gamma_\alpha^\beta \mathfrak{G}^\beta \subset \mathfrak{G}^\alpha.$$

We shall show next that

(8) $$\lambda_\alpha^\beta \Phi^\beta \mathfrak{B}^\beta = \Phi^\alpha \beta_\alpha^\beta \mathfrak{B}^\beta.$$

To do so we shall show that if $b \in b' \in \mathfrak{B}^\beta$, then if b is a continuous cycle, $S_1' \varphi^\beta b$ and $\varphi^\alpha S_1' b$ are homotopic in L^α mod $L^\alpha \cap C^\alpha$, where S_1' is the natural mapping of a geometric realization of K_1^β into a geometric realization of K_1^α which is determined by S_1. If the point p is in B_1^β, then p and $\varphi^\beta p$ are in the closure of some simplex σ of N^β. Hence from the definition of S_1 it is seen that $S_1' p$ and $S_1' \varphi^\beta p$ are in the closure of $S\sigma$. But since $S\sigma$ contains the point $S_1' p$ of B_1^α, the closure of $S\sigma$ contains $\varphi^\alpha S_1' p$. Hence both $\varphi^\alpha S_1' p$ and $S_1' \varphi^\beta p$ are in the closure of $S\sigma$, a simplex of N^α. But since both the points are in L^α, they are in the closure of some simplex of L^α. Also since S_1' maps a point of C^β into one of C^α, and since condition (3) implies that φ^β and φ^α map points of C^β and C^α respectively into points of C^β and C^α, if p is in C^β, then both $\varphi^\alpha S_1' p$ and $S_1' \varphi^\beta p$ are in C^α. It follows that $S_1' \varphi^\beta b$ and $\varphi^\alpha S_1' b$ are homotopic in L^α mod $L^\alpha \cap C^\alpha$. Formula (8) is proved.

Summarizing the discussion of section 3 we obtain the following

Theorem 3. *A permissible mapping of K^β into K^α determines homomorphisms* (5), (6), *and* (7) *which satisfy* (8).

§ 4. Invariants related to Alexandroff's inner Betti groups.

Let A be a closed subset of a normal space R. We shall study R and A using two homology theories, the classical Cech theory and the theory of Alexandroff's inner Betti groups.[2]) For the Cech theory no further restriction is placed on R and A. But for Alexandroff's theory R is locally compact (= bicompact). In the present section the locally compact case is considered.

By a permissible covering of R we mean a covering by the open sets $\{e_i\}$ and $\{g_j\}$ which satisfy the following conditions (9) through (15).

(9) The e_i cover A.

(10) The g_j cover $R - A$.

(11) $g_j \cap A = \emptyset$.[5])

(12) If the elements of any subset of $\{e_i\}$ have a common point in R, they have a common point in A.

(13) If $\bar{e}_i \cap A$ is compact, then $\bar{e}_i$ is compact.

(14) If $e_i \cap g_j \neq \emptyset$ and $\bar{g}_j$ is not compact, then $\bar{e}_i \cap A$ is not compact.

(15) If $e_i \cap g_j \neq \emptyset$, then $\bar{g}_j \cap A \neq \emptyset$.

Theorem 4. *Any covering of R has a refinement that is permissible. Here as throughout the article all coverings are by open sets.*

Proof. Consider any covering of R made up of sets $\{e_i\}$ meeting A and $\{g_j\}$ not meeting A. The covering made up of $\{e_i\}$ and $\{g_j, e_i - A\}$ is a refinement which satisfies (9) through (11). This refinement can be further refined by the methods of [1] to obtain a refinement of the original covering which satisfies (9) through (13).

Assume that the covering consisting of $\{e_i\}$ and $\{g_j\}$ satisfies (9) through (13). Suppose that as an exception to (14) we have $e_i \cap g_j \neq \emptyset$, $\bar{g}_j$ is not compact, and $\bar{e}_i \cap A$ is compact. Then by (13) we see that $\bar{e}_i$ is compact. Hence $\overline{e_i \cap g_j}$ is compact. Since R is locally compact and normal, there is an

[5]) $\emptyset$ denotes the empty set.

open set g such that $g \supset \overline{e_i \cap g_j}$, $\bar{g}$ is compact, and $\bar{g} \cap A = \emptyset$. In the covering considered g_j is deleted and replaced by the two sets $g_j - \bar{e}_i$ and $g_j \cap g$. This replacement gives a refinement of the covering with one less exception to (14). Also this replacement does not introduce any exception to (9) through (13). Thus we get a refinement with no exception to (9) through (14).

Assume that a covering satisfies (9) through (14). Consider an e_i. If $e_i \cap A$ contains no limit point of $e_i - A$, then $e_i \cap A$ and $e_i - A$ are both open; in this case we replace e_i by the two open sets $e_i \cap A$ and $e_i - A$. We get a refinement of the covering such that there is no exception to (15) involving e_i. On the other hand if $e_i \cap A$ contains a limit point of $e_i - A$, we proceed as follows. We consider the sum $\sum' \bar{g}_j$ of those $\bar{g}_j$ for which $\bar{g}_j \cap A = \emptyset$. We replace e_i by the two sets $e_i - \sum' \bar{g}_j$ and $e_i - A$. This gives a refinement of the covering such that there is no exception to (15) involving e_i. Handling all the e_i in the same way we get a refinement of the covering satisfying (9) through (15). Theorem 4 is proved.

A permissible covering consisting of $\{e_i^1\}$ and $\{g_i^1\}$ is said to be a permissible refinement of a permissible covering consisting of $\{e_i^2\}$ and $\{g_j^2\}$ if the first covering is a refinement of the second and each g_j^1 is a subset of some g_j^2.

Theorem 5. *Any two permissible coverings of R have a common permissible refinement.*

Proof. Consider the permissible coverings Ω^k, $k = 1, 2$, consisting of $\{e_i^k\}$ and $\{g_j^k\}$. These two coverings have a common refinement. Hence by theorem 4 they have a refinement Ω^3 that is permissible. Let Ω^3 consist of $\{e_i^3\}$ and $\{g_j^3\}$. Form all possible sets g_j^4 each of which is the intersection of three sets, one of which is a g_j^1, one a g_j^2, and the third a g_j^3. The covering consisting of $\{e_i^3\}$ and $\{g_j^4\}$ is a permissible refinement of Ω^1 and Ω^2.

Let K^β be the nerve of the permissible covering Ω^β. Let L^β be the subcomplex of K^β made up of all the simplexes whose vertices correspond to sets e_i^β. As in [1] let a simplex of K^β be special if each vertex of the simplex corresponds to a set of Ω^β whose closure is not compact. These special simplexes make up the special subcomplex C^β.

Theorem 6. *The complexes K^β, L^β, and C^β satisfy conditions* (2) *and* (3).

Proof. Condition (2) is a consequence of the definition of L^β. Also (3) is a consequence of (14).

If Ω^β is a permissible refinement of Ω^α, there is a projection ω_α^β of Ω^β into Ω^α such that each g_j^β projects into a g_j^α. Such a projection is permissible.

The permissible projection ω_α^β determines a simplicial mapping S_α^β of K^β into K^α. We shall show next that S_α^β satisfies condition (4) and hence is permissible. From the definitions of C^β and L^β it is seen that $S_\alpha^\beta C^\beta \subset C^\alpha$

and $S_\alpha^\beta L^\beta \subset L^\alpha$. Condition (15) implies that B^β is made up of those simplexes of K^β whose vertices correspond to a set J of the g_j^β with the two properties that $\bar{g}_j^\beta \cap A \neq \emptyset$, all g_j^β in J, and there is a point common to some e_i^β and all the g_j^β with j in J. This gives $S_\alpha^\beta B^\beta \subset B^\alpha$. Similarly $S_\alpha^\beta N^\beta \subset N^\alpha$. Finally we observe that R^β is the subcomplex of K^β made up of the simplexes whose vertices correspond to elements of $\{g_j^\beta\}$. But since ω_α^β is permissible, a g_j^β projects into a g_j^α. Hence we have $S_\alpha^\beta R^\beta \subset R^\alpha$. The proof that S_α^β is permissible is complete.

Since S_α^β is permissible, theorem 3 gives the homomorphisms (5), (6), and (7). Using theorem 5 we see that we have inverse spectra $[\mathcal{B}^\beta; \beta_\alpha^\beta]$, $[\mathcal{L}^\beta; \lambda_\alpha^\beta]$, and $[\mathcal{G}^\beta; \gamma_\alpha^\beta]$; in these spectra only permissible projections are admitted. Let the limit groups of these spectra be $\mathfrak{B}$, $\mathfrak{L}$, and $\mathfrak{G}$ respectively. These groups are taken as discrete.

Theorem 7. *We have the isomorphism*

$$\mathfrak{B}/\mathfrak{G} \simeq \mathfrak{L}.$$

Proof. Theorem 7 is a consequence of condition (8) of theorem 3.

Theorem 8. *The group $\mathfrak{L}$ is a subgroup of the inner Betti group of A.*[2])

This theorem follows from the conditions (12) and (13).

§ 5. Invariants related to the Cech homology groups.

In section 4 the local compactness of R was employed only in handling difficulties arising in the consideration of special elements of a covering. In the CECH homology theory elements of a covering are not singled out as special. Hence some of the considerations of section 4 give the following theorem.

Theorem 9. *If as in the* CECH *homology theory no elements of a covering are considered to be special, the three groups appearing in theorem 7 can be defined and the isomorphism of theorem 7 proved on the assumption that R is normal.*

§ 6. The simplicial case.

Let there be a simplicial division of R into a finite complex K such that A carries a subcomplex L and (2) is satisfied. We have for K and L the groups $\mathcal{B}$, etc., of section 2 and the groups of the CECH type $\mathfrak{B}$, etc., of section 5.

Theorem 10. *We have the isomorphisms $\mathcal{B} \simeq \mathfrak{B}$, $\mathcal{G} \simeq \mathfrak{G}$, and $\mathcal{L} \simeq \mathfrak{L}$.*

Proof. Let K_n, $n = 0, 1, 2, \ldots$, denote the n^{th} barycentric subdivision of K with the understanding that $K_0 = K$. Let N_n, B_n, and R_n be defined in K_n as N_1, B_1, and R_1 are defined in K_1. Any simplex of N_i, $i = 0, 1, 2, \ldots$, intersects B_{i+1} in a subcomplex which is the subdivision of a cell x_1, the faces of x_1 being the intersections of B_{i+1} and the faces of the given simplex of N_i. The same simplex of N_i determines in the same way a cell x_2 as its intersection with B_{i+2}. All such x_1 and x_2 determine cell complexes X_1 and X_2 respectively of which B_{i+1} and B_{i+2} are subdivisions. Also X_1 and X_2 are isomorphic under the association of x_1 and x_2. Let $\mathcal{B}^n$ be defined for K_n as $\mathcal{B}^\alpha$ is defined for K^α. It is shown in [2] that the chain mapping $x_2 \to x_1$ determines a correspondence between cycles that leads to the first two of the isomorphisms

$$(16) \qquad \mathcal{B}^{n+1} \simeq \mathcal{B}^{n+2}, \; \mathcal{G}^{n+1} \simeq \mathcal{G}^{n+2}, \; \mathcal{L}^{n+1} \simeq \mathcal{L}^{n+2}.$$

The third isomorphism of (16) is well known. These isomorphisms will prove theorem 10 when we have defined a cofinal sequence of permissible coverings such that the corresponding groups and projections are the groups and isomorphisms of (16).

Let Ω^n, $n = 0, 1, 2, \ldots$, be the covering of R by the open stars of the vertices of K_n. Then K_n can be considered as the nerve of Ω^n. It is seen that each Ω^n is permissible. We shall describe a permissible projection of Ω^{i+2} into Ω^{i+1}, $i = 0, 1, 2, \ldots$, such that the corresponding projection of the nerve K_{i+2} into the nerve K_{i+1} determines a chain mapping of B_{i+2} into B_{i+1} that is homotopic to the chain mapping defined by $x_2 \to x_1$.

Consider a vertex V of B_{i+2}. Let V' be a vertex of B_{i+1} such that V is in the star of V' (in K_{i+1}). Let σ be a simplex of K_i such that V is a vertex of the second barycentric subdivision of the closure of σ. Then since the star of V' contains V, the vertex V' must be in the first barycentric subdivision of the closure of σ. This means that in any permissible projection of Ω^{i+2} into Ω^{i+1}, the corresponding chain mapping induced in the nerves must be such that a vertex of B_{i+2} that is in the cell x_2 is mapped into a vertex of the corresponding x_1. Hence this mapping of the nerves as applied to B_{i+2} is a mapping which is homotopic to the mapping determined by $x_2 \to x_1$. Thus the homomorphisms of the homology groups determined by the permissible projection of Ω^{i+2} into Ω^{i+1} are the isomorphisms (16). Theorem 10 follows.

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