On Schreier extension of rings without zero-divisors.

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1. In the theory of rings an important rôle is played by the extensions of rings. C. J. EVERETT [1][1]) has considered and completely solved the ring-theoretic analogue of O. SCHREIER's extension theory for groups, that is, the following problem: Let R and S be given rings; the problem is to construct all rings T containing an ideal S' isomorphic to S such that

$$T/S' \approx R \tag{1}$$

holds. We call every solution T of this problem a *Schreier extension* (or briefly an extension) *of S by R.*

The present paper gives a necessary and sufficient conditions under which a Schreier extension contains no zero-divisors[2]), resp. a Schreier extension without zero-divisors has a unity. By making use of these theorems we shall get a new proof of a former result of mine stating that an arbitrary ring without zero-divisors has always a minimal extension with unity and without zero-divisors [5].

2. Recently L. RÉDEI has introduced a fundamental method for constructing new structures from given ones. The new structure has been called by him the *skew product* of the given structures [2], [3]. He has also described Schreier's extension theory for rings by making use of the notion of skew product by which one may easily get a survey over all extensions [4]. In what follows we shall use RÉDEI's treatment, by the aid of which EVERETT's extension theorem for rings may be formulated as follows.

Let $R=0, a, b, \ldots$ and $S=\underline{0}, \alpha, \beta, \ldots$ be two arbitrary rings. We consider a Rédeian skew product $T=R\circ S$ of the rings R and S. The elements of T are all pairs (a, α) with $a\in R, \alpha\in S$, further addition and multiplication in T are defined as follows:

$$(a, \alpha)+(b, \beta)=(a+b, [a, b]+\alpha+\beta), \tag{2}$$

$$(a, \alpha)(b, \beta)=(ab, \{a, b\}+\alpha b+a\beta+\alpha\beta) \tag{3}$$

where

$$[a, b], \{a, b\}, \alpha b, a\beta (\in S) \tag{4}$$

[1]) The numbers in brackets refer to the Bibliography at the end of this paper.

[2]) This problem is due to O. STEINFELD.

are functions of two variables with values in S and satisfying the following conditions:

(5) $$[0, a]=[a, 0]=\{a, 0\}=\{0, a\}=a\underline{0}=0\alpha=\underline{0}a=\alpha 0=\underline{0}.$$

Such four functions determine uniquely the skew product $T=R\circ S$. Conversely, by (2), (3) and (5) we obtain

$$(a, \underline{0})+(b, \underline{0})=(a+b, [a, b]),\quad (a, \underline{0})(b, \underline{0})=(ab, \{a, b\}),$$
$$(0, \alpha)(b, \underline{0})=(0, \alpha b),\qquad (a, \underline{0})(0, \beta)=(0, a\beta),$$

therefore the four functions (4) are uniquely determined by the skew product $T=R\circ S$.

The skew product $T=R\circ S$ is a Schreier extension of S by R obeying (2), (3) and (5) if and only if the following conditions are satisfied for all elements $a, b, c\in R$ and $\alpha, \beta, \gamma\in S$:

(6) $$a(\beta+\gamma)=a\beta+a\gamma,\qquad (\alpha+\beta)c=\alpha c+\beta c,$$

(7) $$(a+b)\gamma+[a, b]\gamma=a\gamma+b\gamma,\quad \alpha(b+c)+\alpha[b, c]=\alpha b+\alpha c,$$

(8) $$a\beta\gamma=(a\beta)\gamma,\qquad \alpha\beta c=\alpha(\beta c),$$

(9) $$ab\gamma+\{a, b\}\gamma=a(b\gamma),\qquad \alpha bc+\alpha\{b, c\}=(\alpha b)c,$$

(10) $$(a\beta)c=a(\beta c),$$

(11) $$(\alpha b)\gamma=\alpha(b\gamma),$$

(12) $$\{ab, c\}+\{a, b\}c=\{a, bc\}+a\{b, c\},$$

(13) $$[a, b]=[b, a],$$

(14) $$[a, b]+[a+b, c]=[a, b+c]+[b, c],$$

(15) $$[a, b]c+\{a+b, c\}=[ac, bc]+\{a, c\}+\{b, c\},$$
$$a[b, c]+\{a, b+c\}=[ab, ac]+\{a, b\}+\{a, c\}.$$

These rings T exhaust all Schreier extensions of S by R. The elements $(0, \alpha)$ form an ideal S' of T which is isomorphic to S under the isomorphism $(0, \alpha)\rightarrow\alpha$, further

$$T\,S'\approx R\qquad ((a, \underline{0})+S'\rightarrow a)$$

holds. If the functions (4) satisfy conditions (6)—(15), then the first two of them are called *additive* and *multiplicative factor system*, respectively, while the two last ones are said to be *left* and *right operator set*, respectively. (It is important to emphasize that these operations differ obviously from the usual ones.)

If $[a, b]=\underline{0}$ and $\{a, b\}=\underline{0}$ for all $a, b(\in R)$ then the extensions equivalent to T are said to be *splitting extensions of S by R*. (See [4][14].)

3. In the discussions of Schreier extensions without zero-divisors we shall need the following

Lemma. *Let S be a ring without zero-divisors and T a Schreier extension of S by the ring R. If for the elements $a\neq 0, b\neq 0$ $(\in R)$ we have*

$ab=0$, *then*

(16) $$\{a, b\}+a\beta+\alpha b+\alpha\beta=\underline{0}$$

in S if and only if one of the equations

(16′) $$a\xi=\alpha\xi \quad \text{or} \quad b\xi=\beta\xi \qquad (a\neq 0, b\neq 0)$$

is satisfied by some element $\xi\neq\underline{0}$.

Proof. Let us multiply (16) by $\xi(\neq\underline{0})$ on both sides. Since $ab=0$, by (9), (5) and (11) we have

$$\xi\{a, b\}\xi=\xi(ab\xi+\{a, b\}\xi)=\xi(a(b\xi))=\xi a\cdot b\xi.$$

Therefore, from (16), in view of (11), we get

$$(\xi a+\xi\alpha)(b\xi+\beta\xi)=\underline{0}.$$

The ring S has no zero-divisors, consequently, one of the factors must be zero. This means that one of the equations (16′) has a non-trivial solution, in fact.[3]) One may write α and β instead of $-\alpha$ and $-\beta$ respectively.

The converse of the assertion is clear.

Now we are going to prove the following

Theorem 1. *A Schreier extension T of the ring S by the ring R contains no zero-divisors if and only if S has this property and the equation*

(17) $$a\xi=\alpha\xi \qquad (a\neq 0)$$

admits the only solution $\xi=\underline{0}$.

Proof. In order to prove the necessity of the conditions, let us suppose that the Schreier extension T has no zero-divisors. It is obvious that neither S contains zero-divisors, since S is an ideal in T. On the other hand, if the equation (17) is satisfied by an element $\xi\neq\underline{0}$, then according to

$$(a, -\alpha)(0, \xi)=(0, a\xi-\alpha\xi)=(0, \underline{0})$$

T has zero-divisors. This contradiction implies that the only solution of (17) is $\xi=\underline{0}$, indeed.

Conversely, the stated conditions are sufficient too. Supposing

(18) $$(a, \alpha)(b, \beta)=(0, \underline{0})$$

we shall prove that one of the factors must be zero. (18) is equivalent to

(19) $$ab=0,\ \{a,b\}+a\beta+\alpha b+\alpha\beta=\underline{0}.$$

a) If $a=b=0$, then $\alpha\beta=\underline{0}$ and hence either $\alpha=\underline{0}$ or $\beta=\underline{0}$, i. e. one of the factors in (18) is zero.

b) If in (19_1) just one of the factors vanishes, say, $b=0$, then we get from (19_2)

$$a\beta=(-\alpha)\beta \qquad (a\neq 0).$$

By (17) this means $\beta=\underline{0}$, consequently, one of the factors in the product (18) is zero.

[3]) Since S has no zero-divisors, the equation $r\zeta=\varrho\zeta$ implies $\zeta r=\zeta\varrho$ $(r\in R;\varrho,\zeta\in S)$.

c) If $ab=0$ holds for $a\neq 0$, $b\neq 0$, then, equation (17) having no solution except $\xi=\underline{0}$, the lemma implies

$$(a,\alpha)(b,\beta)=(0,\varrho)\neq(0,\underline{0})\qquad (a\neq 0, b\neq 0, ab=0; \varrho\neq\underline{0}).$$

Consequently, T has no zero-divisors and the proof of Theorem 1 is completed.

As an immediate consequence of Theorem 1 we get the following

Corollary. *A Schreier extension of a ring without zero-divisors by an arbitrary ring contains no zero-divisors if and only if equation* (17) *has no solution other than the trivial one.*

4. In this section we turn our attention to the proof of the following

Theorem 2. *A Schreier extension $T=R\circ S$ without zero-divisors has a unity if and only if R contains a unity e and there exists an element $\eta\in S$ such that*

$$e\xi=\xi-\eta\xi \tag{20}$$

holds for every element ξ of S. Then (e,η) is the unity of T.

Proof. The conditions are necessary. For, let us denote by (e,η) the unity of T and by (a,α) an arbitrary element in T. Then

$$(a,\alpha)(e,\eta)=(ae, \{a,e\}+\alpha e+a\eta+\alpha\eta)=(a,\alpha).$$

Hence it is clear that e must be the unity of R and

$$\{a,e\}+\alpha e+a\eta+\alpha\eta=\alpha. \tag{21}$$

By (5) $a=0$ implies

$$\alpha e+\alpha\eta=\alpha,$$

whence we have (20)[4]. If, however, $a\neq 0$, then from (21) multiplied by an element $\xi(\neq\underline{0})$ on both sides, with regard to (9) and (11), we conclude

$$(\xi a+\xi\alpha)(e\xi+\eta\xi-\xi)=\underline{0}\qquad (a\neq 0, \xi\neq\underline{0}).$$

Since T has no zero-divisors, by Theorem 1 we have

$$e\xi=\xi-\eta\xi.$$

In order to prove the sufficiency of the conditions, let us consider the product

$$(a,\alpha)(e,\eta)=(ae, \{a,e\}+\alpha e+a\eta+\alpha\eta).$$

Introducing the shorter notation

$$\varrho=\{a,e\}+\alpha e+a\eta+\alpha\eta,$$

we have only to verify that $\varrho=\alpha$. Multiplying the last equation by $\xi(\neq\underline{0})$ on both sides, it follows (using (9))

$$\xi\varrho\xi=\xi a(e\xi-\xi+\eta\xi)+\xi\alpha(e\xi+\eta\xi).$$

By (20) we have

$$\xi\varrho\xi=\xi\alpha\xi$$

and this implies, as asserted, that $\varrho=\alpha$ which completes the proof.

[4]) Similarly as in [3]).

5. We are going to give an example for a splitting Schreier extension without zero-divisors.

Let $S = \underline{0}, \alpha, \beta, \ldots$ denote an arbitrary ring without zero-divisors. The ring of rational integers will be denoted by I, and the residue class ring of I mod m by $I/(m)$ where m is a rational integer. Let us denoty be $\bar{0}, \bar{1}, \ldots, \overline{m-1}$ the elements of $I/(m)$, if $m = 0$, we have obviously $\bar{a} = a$. The integers r for which with a suitable element $\varrho (\in S)$

$$r\xi = \varrho\xi = \xi\varrho \qquad (\xi \in S) \tag{22}$$

has a solution $\xi \neq \underline{0}$, form an ideal of I. Let the basis element of this ideal be $m (\geqq 0)$. We construct the splitting Schreier extension T of S by $I/(m)$. The elements of T are of the form $(\bar{a}, \alpha)$ $(\bar{a} \in I/(m), \alpha \in S)$. In the ring T we add and multiply according to the rules

$$(\bar{a}, \alpha) + (\bar{b}, \beta) = (\overline{a+b}, \alpha + \beta),$$
$$(\bar{a}, \alpha)(\bar{b}, \beta) = (\overline{ab}, \bar{b}\alpha + \bar{a}\beta + \alpha\beta)$$

where $\bar{x}\xi = x\xi$, x denoting the least non-negative representative of the residue class $\bar{x}$.

Now it is easy to prove the following result [5]:

The splitting Schreier extension T of S by $I/(m)$ is the minimal[5]*) extension with unity and without zero-divisors of the ring S.*

It is readily seen that the only solution of the equation

$$\bar{a}\xi = \alpha\xi \qquad (\bar{a} \neq 0)$$

is $\xi = \underline{0}$, therefore, by Corollary, T contains no zero-divisors. By Theorem 2 T has a unity, namely $(\bar{1}, \underline{0})$. Finally, the minimality of T follows from the fact that m is the minimal integer ($\geqq 0$) satisfying (22). The proof is thus completed.

Bibliography.

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(Received September 1, 1952.)

[5]) This means that no proper subring of T is a ring with unity containing S.