On algebraically closed abelian groups.

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§ 1. Introduction.

Let A be an additive abelian group. By x, a, b, c we denote group elements, while the other small Latin letters are reserved for elements of an operator ring which, with the exception of § 3, coincides with the ring of rational integers.

An "algebraic" equation with an unknown x in A, constructed by the aid of addition as the only operation defined in A, can always be written, by the commutativity of A, in the form

$$(1) \qquad nx = c.$$

If (1) is solvable in A for every $c \in A$ and any integer $n > 0$, then we call A an algebraically closed (abelian) group in the sense of T. SZELE [6]. This fact is obviously equivalent to $nA = A$ $(n = 1, 2, 3, \dots)$.[1]) A full structural characterization of the algebraically closed abelian groups is given by the well known fact that such a group is a direct sum of rational groups and groups of type (p^∞), and conversely. However, we shall make no use of this fact.

T. SZELE has raised the question whether or not in an algebraically closed (abelian) group A every compatible system of equations in several unknowns admits a solution which lies in A. This question is answered affirmatively in the present note. Moreover, a generalization of this result for abelian groups with operators is given. As an application we obtain a theorem concerning the solvability of a system of linear equations over a skew field, without any restriction on the cardinal number of the equations and of the unknowns.

[1]) The terminology "algebraically closed abelian group" is motived by the analogy between these groups and algebraically closed fields (see [6]). See also footnote[2]) below.

§ 2. Algebraically closed abelian groups.

Let us consider a set of indeterminates x_ν, where ν ranges over an arbitrary set of indices (ordered or not). Further, let $\{f_\mu(x)=c_\mu\}$ be an arbitrary set of equations

$$f_\mu(x)=r_{\mu 1}x_{\nu_1}+r_{\mu 2}x_{\nu_2}+\cdots+r_{\mu k}x_{\nu_k}=c_\mu(\in A), \tag{2}$$

each $f_\mu(x)$ containing only a finite number of terms; the coefficients $r_{\mu i}$ in (2) are rational integers. Clearly, the most general form of a system of algebraic equations in an abelian group A is $\{f_\mu(x)=c_\mu\}$ where the c_μ's are given elements of A. We say that the system $\{f_\mu(x)=c_\mu\}$ is *compatible*, if any identity

$$s_1 f_{\mu_1}+s_2 f_{\mu_2}+\cdots+s_l f_{\mu_l}=0$$

(for a finite number of the f_μ's) implies

$$s_1 c_{\mu_1}+s_2 c_{\mu_2}+\cdots+s_l c_{\mu_l}=0.$$

Now we prove the following

Theorem 1. *If A is an algebraically closed abelian group, then every compatible system $\{f_\mu(x)=c_\mu\}$ of equations in A admits a solution $x_\nu=a_\nu\in A$.* [2])

Proof. Consider a compatible system $\{f_\mu(x)=c_\mu\}$ of equations in an arbitrary algebraically closed abelian group A. Then there exists one (and essentially only one) abelian group G obtained by adjoining to A elements x_ν^* subject to the relations $f_\mu(x^*)=c_\mu$, i. e.

$$r_{\mu 1}x^*_{\nu_1}+r_{\mu 2}x^*_{\nu_2}+\cdots+r_{\mu k}x^*_{\nu_k}=c_\mu . \tag{3}$$

As a matter of fact, the compatibility of the system $\{f_\mu=c_\mu\}$ implies that the group G so defined contains A (itself and not only a homomorphic image of A) as a subgroup. According to a theorem of R. Baer [1], A, as an algebraically closed group, is a direct summand of every containing abelian group:

$$G=A+B.$$

Consequently, every element in G may be represented in one and only one

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[2]) Theorem 1 can be regarded as a further statement justifying the terminology "algebraically closed abelian group". From this point of view the following remark is of some interest. Theorem 1 implies that an abelian group A is algebraically closed if and only if every system of equations which can be solved in some abelian group containing A, can also be solved in A itself. Then one sees that the notion of algebraically closed abelian group is the abelian analogue of the notion of algebraically closed group recently introduced by W. R. Scott [5] and B. H. Neumann [3]. The similarity of these terms involves no confusion, since an algebraically closed group can never be abelian (it is even always an infinite simple group; see [3]). The two concepts are in the same relation as "free group" and "free abelian group".

way in the form $a+b$ with $a \in A$ and $b \in B$. This applies in particular to the elements $x_\nu^* \in G$ so that

$$x_\nu^* = a_\nu + b_\nu \quad (a_\nu \in A, b_\nu \in B).$$

Then (3) implies

$$\begin{aligned} (4) \qquad & r_{\mu 1}(a_{\nu_1} + b_{\nu_1}) + \cdots + r_{\mu k}(a_{\nu_k} + b_{\nu_k}) = \\ & = (r_{\mu 1} a_{\nu_1} + \cdots + r_{\mu k} a_{\nu_k}) + (r_{\mu 1} b_{\nu_1} + \cdots + r_{\mu k} b_{\nu_k}) = f_\mu(a) + f_\mu(b) = c_\mu . \end{aligned}$$

Since c_μ is an element in A, it follows that $f_\mu(b) = 0$; and so the equations (4) show that the elements $x_\nu = a_\nu \in A$ yield a solution of the system $\{f_\mu(x) = c_\mu\}$.

§ 3. Algebraically closed abelian groups with operators.

Now we are going to generalize Theorem 1 for groups with operators. We consider an additive abelian group A admitting a ring R with unit element 1 as a left operator domain. We assume, furthermore, that $1a = a$ for each element $a \in A$. In the case treated in the previous section, R was the ring of rational integers. In accordance with this we use for the notation of elements of R the same symbols (e. g. r, s) as previously for integers. Let L be a left ideal in R. An *operator homomorphism* of L into the group A is a single valued function l^α of the elements l in L with values in A, which satisfies

$$(r_1 l_1 + r_2 l_2)^\alpha = r_1(l_1^\alpha) + r_2(l_2^\alpha); \quad (r_i \in R, \ l_i \in L, \ i = 1, 2).$$

We recall an important result of R. Baer [2] which generalizes Baer's theorem mentioned above:

The abelian group A over a ring R with unit element is a direct summand of every containing abelian group over R if and only if A possesses the following property:

(P): *To every left ideal L in R and to every operator homomorphism $l \to l^\alpha$ of L into A there exists some element a in A such that $l^\alpha = la$ for every l in L.*

Now, generalizing Theorem 1, we can prove, in the same way as in § 2, the following

Theorem 2. *If A is an abelian group over a ring R (with unit element) with the property* (P), *then every compatible system* (2) *of equations in A admits a solution in A.*

According to this theorem the abelian groups over R with the property (P) generalize the concept of algebraically closed abelian groups to the case of groups with operators.

§ 4. Systems of linear equations over a skew field.

A skew field F can be regarded as an additive group F^+ admitting F itself as a left operator domain. Hence, as the group F^+ possesses obviously the property (P) (with $R=F$), Theorem 2 implies the following

Theorem 3. *Every compatible system of linear equations (with an arbitrary cardinal number of the equations and an arbitrary cardinal number of the unknowns) over a skew field F admits a solution in F.*[3])

I owe Professor B. H. NEUMANN the following formulation of Theorem 3: *An arbitrary system of linear equations over a skew field F admits a solution in F if and only if any finite subsystem possesses a solution in F.*

Thus the solvability being a property of "finite character", we have by TUKEY's lemma: *Any system of linear equations over a skew field contains a maximal solvable subsystem.*

For the sake of completeness we give an immediate proof of this Theorem 3, without making use of the theorem of BAER quoted in § 3. By virtue of § 2 it is sufficient to prove the following statement which is of course a special case of the theorem of BAER:

An arbitrary abelian group A over a skew field F is a direct summand of every containing abelian group G over F.

As a matter of fact, let B be a *greatest* (admissible) subgroup of G whose meet with A is 0. The subgroup of G generated by A and B is their direct sum; and hence it suffices to prove that $A+B=G$. For this purpose we consider an arbitrary element $c\neq 0$ of G. The maximal property of B implies

$$(A+B)\cap Fc\neq 0$$

where Fc denotes the (admissible) subgroup of G generated by the element c. Thus a relation

$$sc=a+b\neq 0 \qquad (s\in F, a\in A, b\in B)$$

holds, and consequently

$$c=s^{-1}a+s^{-1}b\in A+B$$

which completes the proof.

Remark. The statements expressed in Theorem 3 and in its above corollary are by no means trivial, as is shown by the following counter example of an ordinary abelian group C (with the ring of integers as operator domain). Let C be an infinite cyclic group generated by the element a. Consider the system of equations

$$x_1+2x_2=a, \quad x_2+2^2x_3=a, \ldots, x_n+2^n x_{n+1}=a, \ldots .$$

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[3]) Of course, however, each single equation contains a finite number of unknowns only.

Every finite subsystem of this system admits obviously a solution in C, although the whole system is not solvable in C. More exactly, a subsystem of this system is solvable in C if and only if it does not contain an infinity of the above equations. Hence the system has no maximal solvable subsystem. (The above system of equations was suggested by a famous group construction due to L. Pontrjagin [4]).

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