On arbitrary systems of linear equations.

By T. SZELE in Debrecen.

In a recent paper [2][1]) S. GACSÁLYI has shown that any compatible system of linear equations (with an arbitrary cardinal number of the equations resp. of the unknowns) over a skew field F admits a solution in F.[2]) [3]) A system of linear equations is said to be compatible if every linear dependence relation for a finite number of linear forms on the left hand sides of the equations is satisfied also by the given elements of F on the corresponding right hand sides. This theorem seems to be new and gives rise to the problem of extending the classical methods for solution of systems of linear equations (see [3], p. 106) from the finite case to the infinite case. The possibility of this extension will be shown in the present note.

Let us consider an arbitrary compatible system

$$(1) \qquad f_\nu(\ldots, x_\lambda, \ldots) = b_\nu \in F$$

of linear equations over a skew field F, where λ and ν run resp. over an arbitrary set of indices (wich is not assumed to be ordered). Here

$$(2) \qquad f_\nu(\ldots, x_\lambda, \ldots) = \sum_\lambda a_{\nu,\lambda} x_\lambda$$

is a linear form of the x_λ's and the coefficients $a_{\nu,\lambda} \in F$ are $= 0$ for all but a finite number of λ. The unknowns x_λ can also be regarded as indeterminates and then the set of all x_λ's generates a module M of linear forms over F. Let N be the submodule of M generated by the f_ν's. The set of all f_ν 's contains, by ZORN's lemma (or the equivalent lemma of TUKEY), a maximal linearly independent subsystem S, and, by the compatibility of the system (1), the set of all solutions of the system $f_\nu = b_\nu (f_\nu \in S)$ coincides with the set

[1]) The numbers in brackets refer to the Bibliography at the end of this note.

[2]) Each single equation, however, contains a finite number of unknowns only.

[3]) This seems to be in contradiction with a statement in [1] (see „Remarque" p. 56). This seeming contradiction arises from the definition of a system of linear equations given in [1] which implies that a system $x_\lambda = \xi_\lambda \in F$ can be a solution only if $\xi_\lambda = 0$ holds for all but a finite number of the λ's.

of all solutions of the whole system (1). Hence we may assume in the sequel that the given system of all linear forms f_ν in (1) is independent.

Now, it is easy to show that there exists a (possibly void) subset $X' = \ldots, x'_\mu, \ldots$ of the x_ν's such that the f_ν's and the elements of X' form a linearly independent generating system (i. e. a basis) of M or, in other words, M is the direct sum of N and the submodule (X') of M generated by X':

$$M = N + (X'). \tag{3}$$

This means that for each indeterminate x_λ an indetity of the form

$$x_\lambda = \sum_{i=1}^{n_\lambda} c_{\lambda\nu_i} f_{\nu_i} + \sum_{j=1}^{m_\lambda} d_{\lambda\mu_j} x'_{\mu_j} \tag{4}$$

$$(c_{\lambda\nu_i} \in F,\ d_{\lambda\mu_j} \in F,\ m_\lambda \text{ and } n_\lambda \text{ are integers} \geqq 0)$$

holds (with use of (2)). Substituting (4) into (2) (as well as (2) into (4)) we obtain identities in the indeterminates x_λ,[4]) and so it is evident that the solutions of the given system (1) coincide with those of the system consisting of all the equations

$$x_\lambda = \sum_{i=1}^{n_\lambda} c_{\lambda\nu_i} b_{\nu_i} + \sum_{j=1}^{m_\lambda} d_{\lambda\mu_j} x'_{\mu_j}. \tag{5}$$

It is clear, however, that *any solution of the system* (5) *can be obtained by substituting for the indeterminates* x'_μ *arbitrary elements of* F, *and by determining the values of the other unknowns* x_λ *from* (5).

Thus we have shown that the classical method for solving systems of linear equations can be extended from the finite to the infinite case. Note that the cardinal number $\mathfrak{f}$ of the unknowns to be "freely chosen" (i. e. the cardinal number of the set X') is given by the rank of the factor module M/N (see (3)). Consequently, the cardinal number $\mathfrak{u}$ of the unknowns and the cardinal number $\mathfrak{e}$ of the equations of the given system (1) (which is assumed to be independent) verify the relation

$$\mathfrak{u} = \mathfrak{e} + \mathfrak{f}$$

in analogy with the finite case.

The above statement implies, in particular, the result of GACSÁLYI and the following corollary to this: the compatible system (1) of linear equations admits exactly one solution if and only if (the set X' is void i. e.) the linear forms $f_\nu(x)$ generate the whole module M of the x_λ's over F.

Finally we prove the validity of (3). Let the system $X' = \ldots, x'_\lambda, \ldots$ be a maximal mod N independent subset of the set of all indeterminates x_λ. [A subset of indeterminates x_λ is called mod N independent if it contains no

[4]) This is assured by the independence of the system $\ldots, f_\nu, \ldots, \ldots, x'_\mu, \ldots$ in M.

finite subset $x_1, \ldots, x_k$ $(k > 0)$ such that a relation $e_1 x_1 + \ldots + e_k x_k \in N$ with $0 \neq e_i \in F (i = 1, \ldots, k)$ holds.] Such a maximal system X' always exists (by TUKEY's lemma) and by definition satisfies (3).

Bibliography.

[1] N. BOURBAKI, Éléments de mathématique. I. Livre II. Algèbre. Chapitre II. Algèbre linéaire. (Paris, 1947.)

[2] S. GACSÁLYI, On algebraically closed abelian groups. *Publ. Math. Debrecen*, **2** (1952) 292—296.

[3] B. L. VAN DER WAERDEN, Moderne Algebra. II. (Berlin, 1940.)

(Received October 14, 1952.)