On non-countable abelian p-groups.

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According to an important theorem of H. PRÜFER a countable abelian p-group is the direct sum of cyclic groups if and only if it does not contain elements of infinite height [4].[1])[2]) It is well-known that this theorem cannot be extended to non-countable groups. The first counter example was given by PRÜFER in [3]. Because of the great importance of this fact it seemed desirable to give „counter examples" as simple as possible, i. e. to construct non-countable abelian p-groups which cannot be decomposed into a direct sum of cyclic groups although they contain no element of infinite height. As a matter of fact, there were given later simpler examples of such groups by H. ULM [5], A. KUROŠ [2], and L. KULIKOV [1]. Now, in the present note I shall give a new proof of the fact that the counter example of KULIKOV has the property mentioned above. My proof seems to be shorter and simpler than the others, and in particular suited for class-room use.

Let $\{a_n\}$ be a cyclic group of order p^n generated by the element a_n. (p denotes an arbitrary fixed prime number.) We consider the complete direct sum of the groups $\{a_1\}, \{a_2\}, \ldots, \{a_n\}, \ldots$ i. e. the set of all vectors

$$b = \langle m_1 a_1, m_2 a_2, \ldots, m_n a_n, \ldots \rangle \tag{1}$$

which are added component-wise. ($0 \leqq m_n \leqq p^n - 1$; $n = 1, 2, 3, \ldots$). The elements of finite order of this group form an abelian p-group G of the power of the continuum which contains no element of infinite height. [Clearly an element (1) is of finite order if and only if there exists a maximal number among the orders of $m_1 a_1, \ldots, m_n a_n, \ldots$ and in this case the order of b coincides with this maximal number. Further, the height of b is given by the maximal non-negative integer h such that p^h divides m_n for $n = 1, 2, 3, \ldots$]

Now we show that G cannot be decomposed into the (discrete) direct sum of cyclic groups. Assume the contrary and denote by A_n the direct sum of the cyclic direct summands of order $\leqq p^n$ in some direct decomposition

[1]) The numbers in brackets refer to the Bibliography at the end of this note.

[2]) An element $a \neq 0$ of an (additive) abelian p-group is said to have the height h (relative G), if the equation $p^n x = a$ is solvable in G for $n \leqq h$, but not for $n > h$. If $p^n x = a$ has a solution $x \in G$ for any natural number n, then $a \neq 0$ is said to be an element of infinite height.

of G into cyclic groups. Since G is the union of the countable chain $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$, it is sufficient to show that each A_n is a finite group. For this implies that G is (finite or) countable which is a contradiction. For an arbitrary element $b \neq 0$ of A_n of order p^k the sum of k and of the height of b (relative G) is obviously $\leqq n$. Hence there cannot exist two distinct elements

$$b' = \langle m_1' a_1, \ldots, m_n' a_n, \ldots \rangle$$
$$b'' = \langle m_1'' a_1, \ldots, m_n'' a_n, \ldots \rangle$$

in A_n such that

$$(2) \qquad m_1' a_1 = m_1'' a_1, \ldots, m_n' a_n = m_n'' a_n$$

(which already implies the finiteness of A_n). Indeed, if two elements $b' \neq b''$ of A_n satisfy the conditions (2), then for the order p^k resp. the height h of the element $b' - b'' \in A_n$ we have

$$k + h > n$$

which is impossible.

Remark. It is easy to see that the group G contains subgroups which are „counter examples" of power $\aleph_1$. In fact, by the above proof, any non-countable subgroup H of G is a counter example for which *the height of each element $c \in H$ relative H is identical with that relative G.* Now, G contains obviously a subgroup H_1 of order $\aleph_1$. (Each subgroup generated by a subset of power $\aleph_1$ in G is such a group H_1.) This can be extended as follows to a subgroup $H \subseteq G$ of the same power which possesses the mentioned property. Determine for any $0 \neq c \in H_1$ an element $x_c \in G$ such that $p^{h_c} x_c = c$, h_c being the height of c relative G. Denote by $\bar{H}_1 = H_2$ the subgroup of G generated by all elements x_c (c runs over the set of all elements $\neq 0$ of H_1). Construct likewise the subgroups $\bar{H}_2 = H_3$, $\bar{H}_3 = H_4, \ldots$ of G. Then the union H of the groups $H_1, H_2, H_3, \ldots$ is a subgroup of G with the desired properties.

Bibliography.

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[4] H. Prüfer, Untersuchungen über die Zerlegbarkeit der abzählbaren primären Abelschen Gruppen. *Math. Zeitschrift* **17** (1923), 35—61.

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