On direct sums of cyclic groups with one amalgamated subgroup.

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§ 1. Introduction.

In what follows we investigate the structure of the (additive) abelian group G generated by the elements $a_1, a_2, a_3, \ldots$ (in an arbitrary cardinal number) subject to the only relations

(1) $$m_1 a_1 = m_2 a_2 = m_3 a_3 = \ldots \; (= c)$$

where the m_ν's donete arbitrary rational integers.[1]) This group can be described, in SCHREIER's terminology, as *the direct sum of the infinite cyclic groups* $\{a_\nu\}$ *with one amalgamated subgroup* (namely $\{c\}$). If we exlude the trivial case in which some $m_\nu = 0$,[2]) the amalgamated subgroup $\{c\}$ is also an infinite cyclic group. If we adjoin to the relations (1) the relation

(2) $$mc = 0 \qquad (m > 0),$$

then the abelian group $H = \{a_1, a_2, a_3, \ldots\}$ is the direct of the *finite* cyclic groups $\{a_\nu\}$ with one amalgamated subgroup. In both cases, the groups G, H turn out to have, in general, many interesting properties. Many remarkable "counter examples", standing in connection with some problems which have played an important rôle in the development of the theory of abelian groups, belong to this category of groups. In particular, the example of FOMIN [2][3]) for a mixed abelian group which does not decompose into the direct sum of a torsion group and a torsion-free group is given as the abelian group generated by the countable set of elements $a_1, a_2, a_3, \ldots$ subject to the relations

(3) $$pa_1 = p^2 a_2 = p^3 a_3 = \ldots \; (= c),$$

and PRÜFER's example [5] for a countable abelian p-group which cannot be

[1]) The letters $x, a, b, \ldots, g$ denote in the sequel elements of groups and the other small Latin letters ordinary integers.

[2]) Then G is obviously a direct sum of finite cyclic groups.

[3]) The numbers in brackets refer to the Bibliography at the end of this paper.

decomposed into a direct sum of directly indecomposable groups, is obtained from the previous group if we adjoin to the relations (3) $pc=0$.[4]) (Here p denotes an arbitrary prime number.) These properties of the groups in question are contained also in the results of this paper.

Clearly it may be assumed that $m_\nu>1$ for each ν in (1). Then the set of elements $a_1, a_2, a_3, \ldots$ is, in both cases mentioned above, a minimal generating system of the groups G, H. Hence the groups G, H possess always minimal generating systems.

We shall obtain the following results. The direct sum of the (infinite or finite) cyclic groups with one amalgamated subgroup can be represented in the form of the direct sum $A+B$ where A is a countable group (or 0) and B is a direct sum of finite cyclic groups (arbitrary in number). The direct sum G of infinite cyclic groups with one amalgamated subgroup contains always elements of infinite order and can be decomposed into a direct sum of cyclic groups if and only if the set of the (positive) integers m_ν is bounded. The torsion subgroup T of G (formed by the elements of finite order of G) is always a direct sum of cyclic groups, while the factor group G/T is a torsion-free group of rank one, i. e. isomorphic to a subgroup of the additive group of all rational numbers. The group G itself is torsion-free if and only if any two m_i, m_j $(i\neq j)$ are relatively prime. G is the direct sum of T and of a subgroup U of G if and only if there exists no prime number any power of which divides some m_ν. — The direct sum H of the finite cyclic groups $\{a_\nu\}$ with one amalgamated subgroup (defined by the relations (1), (2)) can be decomposed into the direct sum of finite cyclic groups if and only if there exists no prime factor of m, any power of which divides some m_ν.

Similar results are obtained, in a more general context, in a paper of HANNA NEUMANN on the direct sum of a finite number of infinite cyclic groups with more amalgamated subgroups ([4], p. 684, 5.7 Theorem).

§ 2. The direct sum of infinite cyclic groups with one amalgamated subgroup.

We consider the abelian groups G generated by the elements $a_1, a_2, a_3, \ldots$ subject to the only relations (1). We assume, as mentioned before, that $m_\nu>1$ for every ν. Clearly c, and consequently each a_ν, is an element of infinite order in G. Now let $S=(a_1, a_2, a_3, \ldots)$ be a subset of the set of all a_ν's such that the corresponding natural numbers $m_1, m_2, m_3, \ldots$ are all dis-

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[4]) Another interesting property of this group is that it has a cyclic subgroup (namely $\{c\}$) such that the corresponding factor group is a direct sum of cyclic groups, although the whole group itself does not decompose into a direct sum of cyclic groups. (See [1] and [3].)

tinct and for each ν the number m_ν is equal to one of the m_k's $(k=1, 2, 3, \ldots)$. The system S is obviously at most countable. Since for each $a_\nu \notin S$ the corresponding m_ν is equal to one (well defined) m_k, for the element $a_\nu - a_k$ we have, by (1),

$$m_k(a_\nu - a_k) = 0. \tag{4}$$

On the other hand, G is generated by the set S and the set of all elements $a_\nu - a_k$ $(a_\nu \notin S)$ the latter being subject only to the relations (4). This shows the validity of the direct decomposition

$$G = A + B \tag{5}$$

where A is the subgroup of G generated by the (finite or countable) set $S = (a_1, a_2, a_3, \ldots)$, and B is the direct sum of the finite cyclic groups $\{a_\nu - a_k\}$ $(a_\nu \notin S)$. Therefore we have

Theorem 1. *An arbitrary abelian group G generated by the elements a_ν subject to the only relations* (1) *can be represented in the form* (5) *where A is a (countable) group of the same kind but with $m_i \neq m_k$ for $i \neq k$ and B is a direct sum of finite cyclic groups (arbitrary in number).*

On account of this theorem we can restrict ourselves to the investigation of the *countable* group $A = \{a_1, a_2, a_3, \ldots\}$ for which $m_i \neq m_k$ $(i \neq k)$ holds in (1).

Theorem 2. *The group G is a direct sum of cyclic groups if and only if the set of the (positive) integers m_ν in* (1) *is bounded.*

As a matter of fact, if the set of the m_ν's is bounded, then the set S in finite and consequently A, as a finitely generated abelian group, is a direct sum of cyclic groups. Thus, by (5), G itself is a direct sum of cyclic groups. Note that exactly one of the cyclic direct summands is infinite, for G does not have two linearly independent elements of infinite order. — On the other hand, if the set of the integers m_ν is not bounded, then the relations (1) show that the equation $nx = c$ is solvable in G for some arbitrary large integers n. Hence, c being an element of infinite order, G cannot be a direct sum of cyclic groups.

Theorem 3. *The torsion subgroup T of G is a direct sum of cyclic groups, and the factor group G/T is a torsion-free group of rank one, more exactly, it is isomorphic to the subgroup of the additive group of all rational numbers generated by the numbers $\frac{1}{m_1}, \frac{1}{m_2}, \ldots$*

On account of the decomposition (5) it is sufficient to prove the validity of Theorem 5 for the case $G = A$. Then T is countable, and so the first statement of Theorem 3 follows, by a well-known theorem of PRÜFER [5], from the fact that G contains no element of prime power order and of infinite height.[5]) Indeed, if g would be an element of order p^k (p prime) and of infinite height, then the equation $p^r x = g$ is solvable in G for any natural number r; consequently the equation $nx = g$ is also solvable in G for any integer $n (\neq 0)$. This means that g is an element of the meet

$$m_1 G \cap m_2 G \cap m_3 G \cap \ldots = \{c\}$$

which is impossible, since the infinite cyclic group $\{c\}$ contains no element $(\neq 0)$ of finite order.

Now, the factor group G/T is a torsion-free group each element $(\neq 0)$ of which has a non-zero multiple $\in \{\bar{c}\}$ where $\bar{c}$ denotes the coset of G (relative to T) containing the element c of G. Therefore G/T is a group of rank one. The correspondence $\bar{a}_\nu \rightarrow \frac{1}{m_\nu}$ induces obviously an isomorphism of the group G/T onto the subgroup of the rational group generated by all numbers $\frac{1}{m_\nu}$.

Theorem 4. *G is torsion-free if and only if any two m_i, m_j $(i \neq j)$ are relatively prime.*

Suppose that any two m_i, m_j $(i \neq j)$ are relatively prime and let $d = r_1 a_1 + \ldots + r_k a_k \notin \{c\}$ be an element of G. We show that d is an element on infinite order in G. Since

$$m_1 m_2 \ldots m_k d = (r_1 m_2 \ldots m_k + \ldots + r_k m_1 \ldots, m_{k-1}) c$$

and c is of infinite order, it suffices to prove the impossibility of the equation

$$r_1 m_2 \ldots m_k + \ldots + r_k m_1 \ldots m_{k-1} = 0. \tag{6}$$

This, however, is obvious. For (6) would imply $m_i | r_i$ $(i = 1, 2, \ldots, k)$ which contradicts our assumption $d \notin \{c\}$.

Conversely, if $m_1 = m_1' t$, $m_2 = m_2' t$, $t > 1$ hold, then the equation

$$m_1 a_1 - m_2 a_2 = t(m_1' a_1 - m_2' a_2) = 0$$

shows that $m_1' a_1 - m_2' a_2 \neq 0$ is an element of finite order in G.

Theorem 5. *The torsion subgroup T of G is a direct summand of G if and only if there exists no prime number any power of which divides some m_ν.*

Let p be a prime number such that an arbitrary power p^n of p divides some m_ν. This means, by Theorem 3, that the factor group G/T contains an infinite chain $\bar{e}_1, \bar{e}_2, \bar{e}_3, \ldots$ of elements of infinite order such that $p\bar{e}_2 = \bar{e}_1$, $p\bar{e}_3 = \bar{e}_2, \ldots$. On the other hand, G contains obviously no such chain. Consequently, a decomposition $G = T + U$ is impossible, for such a decomposition would imply $U \simeq G/T$.

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[5]) The element $g \in G$ of p-power order is said to be of infinite height in G if the equation $p^r x = g$ has a solution in G for any natural number r.

Now we suppose that for each prime p_j there exists an integer n_j ($\geqq 0$) such that $p_j^{n_j}$ divides some m_ν but none of the m_ν's is divisible by $p_j^{n_j+1}$. Then we can determine for each ν a divisor t_ν of m_ν such that each "maximal prime power" $p_j^{n_j}$ divides exactly one number of the system

$$\frac{m_1}{t_1}, \frac{m_2}{t_2}, \ldots, \frac{m_\nu}{t_\nu}, \ldots. \tag{7}$$

Hence we have for the system $a'_\nu = t_\nu a_\nu$ of elements of G the relations

$$\frac{m_1}{t_1} a'_1 = \frac{m_2}{t_2} a'_2 = \ldots (=c) \tag{8}$$

which show, by Theorem 4, that the subgroup $U = \{a'_1, a'_2, \ldots, a'_\nu, \ldots\}$ of G is *torsion-free*. Now, the definition of the number system (7) involves, on account of (8), that the equation $m_\nu x = c$ has a solution $x = b_\nu \in U$ for every ν. By $m_\nu b_\nu = c = m_\nu a_\nu$ any element $a_\nu - b_\nu$ is of finite order. On the other hand, the torsion-free group U and the torsion group V generated by all elements $a_\nu - b_\nu$ generate together the whole group G; consequently the direct decomposition

$$G = U + V$$

holds. The group V coincides obviously with the torsion subgroup T of G.

§ 3. The direct sum of finite cyclic groups with one amalgamated subgroup.

Now we consider the abelian group H generated by the elements a_ν subject to the only relations (1), (2). In the same way as at the beginning of § 1 one can show that $H = A + B$ where A is a countable group of the same kind as H but belonging to a system of pair-wise different m_ν's, and B is a direct sum of finite cyclic groups.

Concerning the question whether or not H can be decomposed into a direct sum of cyclic groups we prove the

Theorem 6. *The abelian group H generated by the elements a_ν subject to the only relations* (1), (2) *is a direct sum of cyclic groups if and only if there exists no prime factor of the number m any power of which divides some m_ν.*

As a matter of fact, suppose that there exists no such prime factor of m. Split each m_ν into the product

$$m_\nu = m'_\nu s_\nu$$

such that any prime factor of m'_ν is a divisor of m while s_ν and m are relatively prime. Then all elements

$$a'_\nu = s_\nu a_\nu$$

subject to the relations

$$m'_1 a'_1 = m'_2 a'_2 = \ldots (= c)$$

generate a subgroup C of H. The order of each element in C is divisible only by such primes which are divisors of m, and, by our assumptions on the numbers m_ν, the orders of all elements in C form a bounded set. Hence C is a direct sum of cyclic groups.

On the other hand, c being an element of order m, there exists, by $(m, s_\nu) = 1$, a multiple c_ν of c such that

$$s_\nu c_\nu = c.$$

Hence we have

$$m_\nu a_\nu = s_\nu m'_\nu a_\nu = c = s_\nu c_\nu$$

i. e.

$$s_\nu (m'_\nu a_\nu - c_\nu) = 0.$$

Thus the subgroup D of H generated by all elements $m'_\nu a_\nu - c_\nu$ is the direct sum of the cyclic groups $\{m'_\nu a_\nu - c_\nu\}$. Further, since the order of no element of D is divisible by a prime factor of m, the groups C and D have no element ($\neq 0$) in common. Finally the direct sum $C + D$ exhausts the whole group H since the set of all elements $a'_\nu = s_\nu a_\nu$ and $m'_\nu a_\nu - c_\nu$ generates, by $(m'_\nu, s_\nu) = 1$, obviously H. Thus we have shown that in this case H is a direct sum of cyclic groups.

Suppose now, conversely, that there exists a prime factor p of m any power of which divides some m_ν. This means that the equation $p^n x = rc$ is solvable in H for any natural number n, rc being an element of p-power order of the cyclic group $\{c\}$. Consequently the group H (containing an element of prime power order and of infinite height) cannot be decomposed into a direct sum of cyclic groups. This completes the proof.

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