On the smallest distance of two lines in 3-space.

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In several introductory textbooks on analytic geometry the theorem on the normal transversal of a pair of straight lines in 3-space is treated with the vector method.[1]) In most cases there is made use of the rules for obtaining by differentiation the extremal values of functions of several variables, which, however, are hardly known to the beginner, acquainted only with quite elementary material. In this little note, it is our aim to show for classroom use that the problem can be treated also without this analytical apparatus. Indeed, we prove using only the elements of vector algebra (the scalar product included) the following

Theorem. *To any two non-parallel lines in 3-space there exists a uniquely determined line intersecting both of them at right angles. The segment of this line between the two points of intersection is shorter than the corresponding segment of any other intersecting line.*

In what follows, bold face type will serve to denote vectors, while scalars will be printed in italic type. $\mathbf{a}\cdot\mathbf{a}=\mathbf{a}^2$, $\mathbf{a}\cdot\mathbf{b}$ denote scalar products.

The parametric equations of two straight lines are

$$\mathbf{r}=\mathbf{a}+t\mathbf{b}, \qquad \mathbf{r}'=\mathbf{a}'+t'\mathbf{b}' \tag{1}$$

respectively, where the vectors

$$\mathbf{b}\neq 0, \qquad \mathbf{b}'\neq 0 \tag{2}$$

are not parallel. Let $\mathbf{e}$ be a unit vector perpendicular to both $\mathbf{b}$ and $\mathbf{b}'$. If for some values t, t' the vector $\mathbf{r}-\mathbf{r}'$ is perpendicular to both lines, then an equation

$$\mathbf{r}-\mathbf{r}'=\mathbf{a}-\mathbf{a}'+t\mathbf{b}-t'\mathbf{b}'=d\mathbf{e} \tag{3}$$

holds with a suitable scalar d. From (3) follows, by scalar multiplication with the unit vector $\mathbf{e}$, that

$$d=(\mathbf{a}-\mathbf{a}')\mathbf{e}. \tag{4}$$

[1]) See e. g.: L. BIEBERBACH, Analytische Geometrie (Berlin, Leipzig, 1932), pp. 81—82. — G. KOWALEWSKI, Einführung in die analytische Geometrie (Berlin, Leipzig, 1923), pp. 109—114. — R. ROTHE, Höhere Mathematik, Vol. 2 (Leipzig, 1951), pp. 191—192.

We show that there exists a uniquely determined pair of real numbers t, t' for which (3) holds with the value (4) for d. By (4), (3) can be written in the form

$$\mathbf{v} = ((\mathbf{a}-\mathbf{a}')\mathbf{e})\mathbf{e}-(\mathbf{a}-\mathbf{a}') = t\mathbf{b}-t'\mathbf{b}'. \tag{5}$$

According to this $\mathbf{v}\mathbf{e}=0$, and the vector $\mathbf{v}$ is parallel to the plane determined by the vectors $\mathbf{b}$ and $\mathbf{b}'$. This plane, however, is spanned by the vectors $\mathbf{b}$ and $\mathbf{b}'$, so that, by virtue of (5), our previous statement becomes obvious.

In order to prove the second assertion of the theorem, we choose the spanning vectors $\mathbf{a}$, $\mathbf{a}'$ in such a way, that their endpoints determine a line perpendicular to both lines considered. Then

$$(\mathbf{a}-\mathbf{a}')\mathbf{b} = (\mathbf{a}-\mathbf{a}')\mathbf{b}' = 0 \tag{6}$$

holds. By (6), we have for the square of the distance between two arbitrary points of the two lines (1):

$$(\mathbf{r}-\mathbf{r}')^2 = (\mathbf{a}-\mathbf{a}')^2+(t\mathbf{b}-t'\mathbf{b}')^2+2(\mathbf{a}-\mathbf{a}')(t\mathbf{b}-t'\mathbf{b}') = \\ = (\mathbf{a}-\mathbf{a}')^2+(t\mathbf{b}-t'\mathbf{b}')^2.$$

This takes on its minimal value (i. e. the value $(\mathbf{a}-\mathbf{a}')^2$) if and only if $t\mathbf{b}-t'\mathbf{b}'=0$, i. e. (by (2)) $t=t'=0$. Thus the theorem is proved.

Remark. The above proof is valid not only for 3-space, but also for n-space; in the case $n>3$, however, we must stipulate that the two lines be not intersecting. For, in the contrary case, there is an infinity of lines passing through the point of intersection and perpendicular to both lines considered. The two non-parallel, non-intersecting lines determine a 3-subspace, and the unit vector $\mathbf{e}$ of the above proof is to be chosen in this subspace.

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