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Gemini functional equations on quasigroups

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1. Introduction

An interesting class of quasigroup functional equations (or identities, or laws) is the class of quadratic equations. Quadratic equation is one in which each variable appears exactly twice. The general study of quadratic equations was initiated by A. Krapež in [5] (they were called 'strictly quadratic' there).

Examples of quadratic equations/identities are:

$x \cdot xy = y$	(Sade's left 'keys' law)
$x \cdot yx = y$	(right semisymmetry)
$xx = yy$	(unipotency)

and the whole class of balanced identities such as:

$x = x$	(trivial identity)
$xy = yx$	(commutativity)
$x \cdot yz = xy \cdot z$	(associativity)
$xy \cdot uv = xu \cdot yv$	(bisymmetry, mediality)
$xy \cdot z = xz \cdot y$	(right permutability)
$x \cdot yz = z \cdot yx$	(Abel-Grassman's law)

as well as many others (see for example [2]).

In general, a balanced equation is one in which each variable appears precisely once on both sides. More on balanced quasigroups can be found in [6] and [7].

To describe gemini equations we first need some definitions.

With every quasigroup $\cdot$ (base set S assumed to be a fixed nonempty set), five more quasigroups, so called parastrophes of $\cdot$ are implicitely given. They are defined by:

$x \cdot y = z$ iff $y * x = z$ iff $x \backslash z = y$ iff $z \backslash\backslash x = y$ iff $z/y = x$ iff $y//z = x$ and have an important role in the description of solutions of quadratic equations. $*$ is usually called the dual of $\cdot$. $\backslash$ and $/$ are the left and right division respectively. $\backslash\backslash$ and $//$ are the duals of $\backslash$ and $/$.

We will use the following notation : $x \circ y$ for xy (i.e. $x \cdot y$) or $x * y$ (i.e. yx) and $x \circ y \circ z$ for $(x \circ y) \circ z$. Therefore $\circ$ is not an operation but just a convenient notation.

If uv is a subterm of a term t then we say that u and v are companions (in t).

The content $\langle u \rangle$ of a quasigroup term u is the set of variables which appear in u. Furthermore we distinguish between linear and quadratic content of u. The linear content $\langle u \rangle_1$ of u is the set of variables which appear precisely once in u . Predictably, the quadratic content $\langle u \rangle_2$ of u is the set of variables appearing exactly twice in u .

Definition 1. The subterms u_1 and u_2 appearing in the equation $w_1 = w_2$ are said to be twins in $w_1 = w_2$ if $\langle u_1 \rangle_1 = \langle u_2 \rangle_1 \neq \emptyset$ and neither is a subterm of the other. We say that u_1 is a twin if there exists a subterm u_2 (of w_1 or w_2) such that u_1 , u_2 are twins or if $\langle u_1 \rangle_1 = \emptyset$.

Definition 2. The quadratic equation $w_1 = w_2$ is said to be gemini if for every subterm uv of w_1 or w_2 which is not a twin and for which $\langle u \rangle_1 \neq \emptyset$, $\langle v \rangle_1 \neq \emptyset$, there exists twins $s_1, s_2, \ldots, s_n$ with $\langle s_n \rangle_1 = \langle u \rangle_1$ or $\langle s_n \rangle_1 = \langle v \rangle_1$ and $\langle s_i \rangle \cap \langle u \rangle = \langle s_i \rangle \cap \langle v \rangle = \emptyset$ for $0 < i < n$ such that $\langle t \rangle_1 = \langle u \rangle_1$ when $\langle s_n \rangle_1 = \langle v \rangle_1$ or $\langle t \rangle_1 = \langle v \rangle_1$ when $\langle s_n \rangle_1 = \langle u \rangle_1$, where $t = uv \circ s_1 \circ \cdots \circ s_n$.

Among equations given above, gemini are : trivial identity, commutativity, Sade's left 'keys' law, right semisymmetry and unipotency.

2. Quadratic equations and cubic graphs

The tools used in this section are essentially due to S. Krstić and first appeared in his PhD thesis [1]. Unfortunately this work is only available in Serbocroatian and for this reason we have given a brief resume of the relevant results.

For every quadratic equation E we define a cubic graph $\Gamma(E)$. First, we replace all occurrences of a quasigroup operation by new binary symbols $V_1, V_2, \ldots, V_n$ so that we can distinguish between them. These will be the vertices of $\Gamma(E)$. If E is the equation $w_1 = w_2$, the edges of $\Gamma(E)$ will be the subterms of w_1 and w_2 . If $t_1 \cdot t_2$ is a subterm of w_1 or w_2 then the corresponding vertex V_i will be incident to edges t_1 , t_2 , $t_1 \cdot t_2$ and no other.

The main operations, say V_p and V_q are incident to the same edge denoted by both w_1 and w_2 . If xx (for some variable x) is a subterm of w_1 or w_2 then x is a loop (circular edge) at the corresponding vertex V_i . As the operation $\cdot$ is binary, every vertex is incident to exactly three edges (a loop being counted twice). So the graph $\Gamma(E)$ is cubic. Example:

Let us write the associativity equation as: $xV_1(yV_2z) = (xV_3y)V_4z$. Then the corresponding graph is:

Figure 1.

Conversely, every cubic graph defines a quadratic equation (which is not unique).

A bridge in a graph is any edge whose removal disconnects the graph. Further, two edges constitute a bridge-couple if neither of them is a bridge and the removal of both disconnects the graph.

A cubic graph Γ is tree-like if every edge of Γ is a bridge or a loop. Γ is tree-like iff it can be obtained from a tree by adjoining a loop to every extremal vertex in the tree.

Connectivity $c(\Gamma)$ of the graph Γ is the minimal number of edges of Γ whose removal disconnects Γ. In a cubic graph we have $c(\Gamma) \leq 3$. $c(\Gamma) = 1$ iff Γ has a bridge and $c(\Gamma) = 2$ iff Γ has no bridges but has a bridge-couple. Note that every cubic graph which is not tree-like and has a bridge has a bridge-couple as well.

Theorem 1 (Menger, see [3])**.** *Any two vertices of a graph Γ can be joined by at least $c(\Gamma)$ arcs such that any two of these arcs have null-dimensional intersection.*

Definition 3. For vertices A and B of a cubic graph Γ $A \sim B$ iff A and B can be joined in Γ by three arcs with disjoint interiors (i.e., having pairwise null-dimensional intersection).

The following two are extreme cases.

- Any two vertices of Γ are $\sim$-equivalent iff $c(\Gamma) = 3$ i.e. Γ has no bridge-couples.
- If A and B are vertices of a tree-like graph Γ then $A \sim B$ iff $A = B$.

We define Γ to be indecomposable iff either Γ is tree-like or $c(\Gamma) = 3$. The other cubic graphs, decomposable ones, we decompose in a following way. Every such graph has a bridge-couple. So let $\{x, y\}$ be a bridge-couple in Γ and Γ'_1, Γ'_2 be the components of $\Gamma \setminus \{x, y\}$. Let $\Gamma_i (i = 1, 2)$ be the graph obtained from Γ'_i by introducing a new edge z_i which connects the endpoints of x and y which belong to Γ'_i .

We say that Γ is a connected sum of Γ_1 and Γ_2 . $A \sim B$ in some of $\Gamma_i (i = 1, 2)$ iff $A \sim B$ in Γ .

Figure 2.

Lemma 1 (Krstić). *Every cubic graph Γ is a connected sum of its indecomposable components $T_1, \ldots, T_n, \Gamma_1, \ldots, \Gamma_m$. $T_1, \ldots, T_n$ are tree-like (with $\sim$-classes singletons) while all $\Gamma_1, \ldots, \Gamma_m$ are $\sim$-classes.*

Lemma 2 (Krstić). *For a cubic graph Γ with more than two vertices, $c(\Gamma) = 3$ iff tetrahedron (i.e. graph usually denoted by K_4) is (homeomorphically) embeddable in Γ .*

Let us call $\sim$-classes with one or two elements small and those with more than two (i.e., at least four) — big. Then we have:

Theorem 2 (Krstić, Krapež [8]). *For a quadratic equation E and associated graph $\Gamma(E)$ the following is equivalent:*
- *a quasigroup satisfying E is isotopic to a group*
- *there is a big $\sim$-class in $\Gamma(E)$*
- *tetrahedron is embeddable in $\Gamma(E)$.*

3. Gemini equations

Lemma 3. *Let t_1, t_2 be subterms of w_1 and/or w_2 such that neither $\{t_1, t_2\} = \{w_1, w_2\}$ nor both t_1, t_2 are variables. If they are twins in E (i.e. $w_1 = w_2$) then $\{t_1, t_2\}$ is a bridge-couple in $\Gamma(E)$.*

Proof. Since not both t_1, t_2 are variables, there are operation symbols in t_1, t_2 . Therefore the set of vertices associated with t_1 and t_2 is not empty. Any subterm of w_1, w_2 containing only variables from $\langle t_1\rangle \cup \langle t_2\rangle$ is a subterm of either t_1 or t_2. Any other subterm of w_1 or w_2 either contains no variables from $\langle t_1\rangle \cup \langle t_2\rangle$ or contains t_1, t_2 or both.

Therefore $\Gamma(E)$ can be split into two subgraphs, the one with vertices/operations from t_1, t_2, the other with the rest of them, such that the only edges connecting the two subgraphs are t_1 and t_2. So $\{t_1, t_2\}$ is a bridge-couple in $\Gamma(E)$. $\square$

Theorem 3. *Quadratic equation E is gemini iff all $\sim$-classes in $\Gamma(E)$ are small.*

Proof. $\Rightarrow$) Using Lemma 1 we can assume that $\Gamma(E)$ is indecomposable and therefore that it has no bridge-couples. If $\Gamma(E)$ is tree-like then all $\sim$-classes are small. So we assume that $\Gamma(E)$ is a $\sim$-class. Assume also that E (i.e. $w_1 = w_2$) has at least three operations and that w_1 has no fewer operations than w_2.

Look at the subterm $V_1(x, y)$ of w_1 (x, y variables).

(a) $V_1(x, y)$ is a twin.

(a1) $x \equiv y$. Then $\{V_1\}$ is a $\sim$-class contrary to our assumption.

(a2) x and y are different variables.

Then there is a subterm t of either w_1 or w_2 which is a twin to $V_1(x, y)$. Consequently $\langle t\rangle_1 = \{x, y\}$.

It cannot be that $V_1(x, y) = t$ is the given equation (w_1 has more then one operation symbol). Therefore by lemma 3 $\Gamma(E)$ has a bridge-couple and is decomposable, contrary to our assumption.

(b) $V_1(x, y)$ is not a twin.

Then there exists twins $s_1, \dots, s_n$ with $\langle s_n\rangle_1 = \{x\}$ or $\langle s_n\rangle_1 = \{y\}$, say the latter and $x, y \notin \langle s_i\rangle$ for $0 < i < n$ such that $\langle V_1(x, y) \circ s_1 \circ \cdots \circ s_n\rangle_1 = \{x\}$. If s_n is not y then y and s_n are twins and $\Gamma(E)$ is decomposable which is a contradiction. Therefore $s_n \equiv y$. Denote the main operation symbol of $V_1(x,y) \circ s_1$ by V_2. As $V_1 \sim V_2$, V_1 and V_2 can be joined by three arcs with disjoint interiors. One such arc consists of the edge $V_1(x,y)$ only. The other two should contain edges x and y respectively. But since $x, y \notin \langle s_i \rangle \quad (0 < i < n)$, both will contain $V_1(x,y) \circ s_1 \circ \cdots \circ s_{n-1}$ and so cannot have disjoint interiors. This is impossible by our assumption that $\Gamma(E)$ is a $\sim$-class.

$\Leftarrow$) For the converse we assume that E is a quadratic equation with all $\sim$-classes of $\Gamma(E)$ small. We should prove that E is gemini.

Let E denote quadratic equation $w_1 = w_2$. Take a subterm uv (assume of w_1) which is not a twin and such that $\langle u \rangle_1 \neq \emptyset$ and $\langle v \rangle_1 \neq \emptyset$.

If $uv \equiv w_1$ then uv is a twin, which is impossible. So there is at least one companion s_1 to uv. Let s_2 be a companion to $uv \circ s_1$ (if it exists). Take $s_3, s_4, \ldots$ similarly so that $w_1 = uv \circ s_1 \circ \cdots \circ s_m$ for some positive m.

We shall distinguish the following cases:

(1) $\langle u \rangle_1 = \langle v \rangle_1$
(2) $\langle u \rangle_1 \setminus \langle v \rangle_1 \neq \emptyset$ and $\langle v \rangle_1 \setminus \langle u \rangle_1 \neq \emptyset$
(3) $\langle u \rangle_1$ is a proper subset of $\langle v \rangle_1$
(4) $\langle v \rangle_1$ is a proper subset of $\langle u \rangle_1$.

(1) If $\langle u \rangle_1 = \langle v \rangle_1$ then $\langle uv \rangle_1 = \emptyset$ and uv is a twin contrary to our assumption.

(2) Two cases are possible:
  (a) for all $i \leq m$ $\langle s_i \rangle \cap \langle u \rangle = \langle s_i \rangle \cap \langle v \rangle = \emptyset$
  (b) there is an $n \leq m$ such that $\langle s_n \rangle \cap \langle u \rangle \neq \emptyset$ or $\langle s_n \rangle \cap \langle v \rangle \neq \emptyset$.

(2a) Let x be a variable from $\langle u \rangle_1$ which is not in $\langle v \rangle_1$ and y a variable in $\langle v \rangle_1$ which is not in $\langle u \rangle_1$. Let also p be the least subterm of w_2 containing both x and y.

Using our convention about names of vertices of $\Gamma(E)$ we shall take terms uv and $V_1(u,v)$ as sinonimous. Also $p = V_2(q,r)$ for some terms q and r. We shall assume $x \in \langle q \rangle$ and $y \in \langle r \rangle$. $\langle V_1(u,v) \rangle_1 \neq \langle V_2(q,r) \rangle_1$, otherwise $V_1(u,v)$ i.e., uv is a twin. Consequently there is a variable z which belongs to one of $\langle V_1(u,v) \rangle_1$, $\langle V_2(q,r) \rangle_1$ but not both. Assume $z \in \langle V_1(u,v) \rangle_1$. Further assume that $z \in \langle v \rangle_1$. This possibility is described by the following form of the equation E:

$$w_1[V_1(u[x], v[V_3[y,z]])] = w_2[V_4[V_2(q[x], r[y]), z]].$$

So $V_1 \sim V_2 \sim V_3 \sim V_4$ which is impossible since all $\sim$-classes are small.

Similar proof can be constructed in the case $z \in \langle u \rangle_1$.

If z is a variable from $\langle V_2(q,r)\rangle_1$ but not from $\langle V_1(u,v)\rangle_1$ we have either:

$$w_1[V_1(u[x],v[y])] = w_2[V_4[V_2(q[x],r[V_3[y,z]]),z]] \quad \text{or}$$
$$w_1[V_4[V_1(u[x],v[y]),z]] = w_2[V_2(q[x],r[V_3[y,z]])]$$

($z \in \langle r\rangle_1$ assumed in both cases).

Then again $V_1 \sim V_2 \sim V_3 \sim V_4$ which is a contradiction.

(2b) The case where both $\langle s_n\rangle \cap \langle u\rangle \neq \emptyset$ and $\langle s_n\rangle \cap \langle v\rangle \neq \emptyset$ is impossible. The proof is analogous to (2a). Therefore we assume that $\langle s_n\rangle \cap \langle u\rangle = \emptyset$ and $\langle s_n\rangle \cap \langle v\rangle \neq \emptyset$. We shall prove $\langle s_n\rangle_1 = \langle v\rangle_1$.

(2b1) $y \in \langle s_n\rangle_1 \cap \langle v\rangle_1$ and there is another variable y' such that $y' \in \langle v\rangle_1$, $y' \notin \langle s_n\rangle_1$. We have:

$$w_1[V_3[V_1(u[x],v[V_2[y,y']]),s_n[y]]] = w_2.$$

Two of the three terms $x, y', V_3(\ldots,s_n)$ define one further operation V_4 and $V_1 \sim V_2 \sim V_3 \sim V_4$ which is a contradiction.

(2b2) $y \in \langle s_n\rangle_1 \cap \langle v\rangle_1$ and there is another variable y' such that $y' \in \langle s_n\rangle_1$ and $y' \notin \langle v\rangle_1$. E then becomes:

$$w_1[V_3[V_1(u[x],v[y]),s_n[V_2[y,y']]]] = w_2.$$

Two of the three terms $x, y', V_3(\ldots,s_n)$ define one further operation V_4 and $V_1 \sim V_2 \sim V_3 \sim V_4$ which is a contradiction.

The only remaining case is:

(2b3) $\langle v\rangle_1 = \langle s_n\rangle_1$.
Let z be a variable which belongs to $\langle s_i\rangle_1$ $(1 < i < n)$ and no other set in the sequence $\langle s_1\rangle_1,\ldots,\langle s_{n-1}\rangle_1$. We have:

$$w_1[V_3[V_2[V_1(u[x],v[y]),s_i[z]],s_n[y]]] = w_2.$$

Two of the three terms $x, z, V_3(\ldots,s_n)$ define one further operation V_4 and $V_1 \sim V_2 \sim V_3 \sim V_4$ which is a contradiction. Therefore z must belong to two sets in the sequence $\langle s_1\rangle_1,\ldots,\langle s_{n-1}\rangle_1$ and consequently $\langle uv \circ s_1 \circ \cdots \circ s_n\rangle_1 = \langle u\rangle_1$.

Assume that there is a term s_i $(1 < i < n)$ and variables $z, z' \in \langle s_i\rangle_1$ such that $z \in \langle s_j\rangle_1$ and $z' \in \langle s_k\rangle_1$ $(j \neq k, j < n, k < n)$. E becomes:

$$w_1[V_3[V_k[V_j[V_i[V_1(u[x],v[y]),s_i[z,z']],s_j[z]],s_k[z']],s_n[y]]] = w_2.$$

It follows that $V_i \sim V_j \sim V_k$ which is a contradiction.

Therefore all variables from $\langle s_i\rangle_1$ also appear in a single s_j which then must be a twin to s_i.

It follows that E is gemini.

To conclude the proof in the case (2b) we should note that the proof of the subcase $\langle s_n\rangle \cap \langle v\rangle = \emptyset$, $\langle s_n\rangle \cap \langle u\rangle \neq \emptyset$ is analogous to the one just given.

It follows that every case is either impossible or implies that E is gemini.

(3) $\langle u\rangle_1$ is a proper subset of $\langle v\rangle_1$.

Two cases are possible:

(a) for all $i \leq m \quad \langle s_i\rangle \cap \langle u\rangle = \langle s_i\rangle \cap \langle v\rangle = \emptyset$

(b) there is an $n \leq m$ such that $\langle s_n\rangle \cap \langle u\rangle \neq \emptyset$ or $\langle s_n\rangle \cap \langle v\rangle \neq \emptyset$.

(3a) Let $x \in \langle u\rangle_1$ and $y \in \langle v\rangle_1 \setminus \langle u\rangle_1$.
Equation E is:

$$w_1[V_1(u[x], v[x,y])] = w_2[y].$$

If $\langle uv\rangle_1 = \langle V_1(u,v)\rangle_1 = \{y\}$ then uv and y are twins which is impossible. Therefore there is a subterm $t = V_3(p,q)$ of w_2 which contains all variables from $\langle V_1(u,v)\rangle_1$. Let $y \in \langle p\rangle_1$ and $y' \in \langle uv\rangle_1$, $y' \in \langle q\rangle_1$. If $\langle t\rangle_1 = \langle V_1(u,v)\rangle_1$ then t and uv are twins which is impossible. Therefore $\langle V_1(u,v)\rangle_1$ is a proper subset of $\langle t\rangle_1$. Let $z \in \langle t\rangle_1 \setminus \langle V_1(u,v)\rangle_1$ and assume $z \in \langle q\rangle_1$. Depending on whether $z \in \langle w_1\rangle$ or not, equation E becomes either:

$$w_1[V_6[V_1(u[x], v[x,y,y']), z]] = w_2[V_3(p[y], q[V_5[y', z]])] \quad \text{or}$$
$$w_1[V_1(u[x], v[x,y,y'])] = w_2[V_6[V_3(p[y], q[V_5[y', z]]), z]].$$

In both cases (and irrespectively of the position of y' in v) $V_3 \sim V_5 \sim V_6$ contrary to our assumption that all $\sim$-classes in $\Gamma(E)$ are small. Therefore the case (3a) is impossible.

(3b) There is an $n \leq m$ such that $\langle s_n\rangle \cap \langle u\rangle \neq \emptyset$ or $\langle s_n\rangle \cap \langle v\rangle \neq \emptyset$.

The proof of this case is analogous to (2b).

Therefore every subcase of (3) is either impossible or else implies that E is gemini.

(4) This case is 'dual' to (3) and can be proved analogously. □

The following theorem shows that quadratic equations are either gemini or force operations satisfying them to be group isotopes.

Theorem 4. *Every quasigroup which satisfies a quadratic equation which is not a gemini equation is isotopic to a group.*

Proof. Directly from Theorem 3. □

Gemini equations can be characterized as quadratic consequences of total symmetry and loop properties :

Theorem 5. *Every gemini equation is satisfied by all totally symmetric loops.*

Proof is by the induction on the number of quadratic variables of an equation.

For $n = 0$, equation is balanced. The statement is true since all Belousov equations are satisfied by all comutative quasigroups (see [4]).

Assume now that all gemini equations with less than n quadratic variables are satisfied by all TS loops. We attempt to prove that this is also true for an arbitrary equation with n quadratic variables.

Let gemini equation $w_1 = w_2$ be given with exactly n quadratic variables. Let uv be a subterm of w_1 or w_2 such that $x \in \langle u\rangle_1$, $x \in \langle v\rangle_1$ and $\langle u\rangle_2 = \langle v\rangle_2 = \emptyset$.

(a) u is not a product. Then $u \equiv x$.

(a1) v is a twin. $x \in \langle v\rangle_1$ so $\langle v\rangle_1 \neq \emptyset$. There is a w — the twin to the v, $x \in \langle v\rangle_1 = \langle w\rangle_1$ so x is a subterm of w. uv is not a subterm of w since $x \notin \langle uv\rangle_1$. So $w \equiv u \equiv x$. But then $\langle v\rangle_1 = \langle w\rangle_1 = \{x\}$ and $v \equiv x$ as well.

Replacing $uv \equiv x^2$ by the unit of a TS loop leads to the gemini equation with less than n quadratic variables.

(a2) v is not a twin.
Then $v \equiv v_1 \circ v_2$. Assume $x \in \langle v_1\rangle$. Since v is not a twin, there are terms $s_1, \ldots, s_m$ such that $\langle s_m\rangle_1 = \langle v_1\rangle_1$ or $\langle s_m\rangle_1 = \langle v_2\rangle_1$ and either $\langle v \circ s_1 \circ \cdots \circ s_m\rangle_1 = \langle v_2\rangle_1$ or $\langle v \circ s_1 \circ \cdots \circ s_m\rangle_1 = \langle v_1\rangle_1$

(a2.1) $\langle s_m\rangle_1 = \langle v_1\rangle_1$.
Since $x \in \langle v_1\rangle_1$, $x \in \langle s_m\rangle_1$ and consequently $s_m \equiv u \equiv x$, $m = 1$ and $v_1 \equiv x$. But then $uv = x(x \circ v_2)$ which is equal to v_2 in TS loops.

(a2.2) $\langle s_m\rangle_1 = \langle v_2\rangle_1$.
Then $\langle v \circ s_1 \circ \cdots \circ s_m\rangle_1 = \langle v_1\rangle_1$. $x \in \langle v\rangle_1$ and $x \in \langle s_1\rangle_1 = \langle u\rangle_1$ so it cannot be $x \in \langle v \circ s_1 \circ \cdots \circ s_m\rangle_1$. This contradiction shows that the case (a2.2) is impossible.

(b) u is a product. Then x has a companion t in u.

(b1) $x \circ t$ is a twin.
Let w be a twin to $x \circ t$. $\langle w\rangle_1 = \langle x \circ t\rangle_1$ and $x \in \langle w\rangle$ so w is a subterm of v. But then there are no quadratic variables in either w or t and $\langle w\rangle = \{x\} \cup \langle t\rangle$.

If t is variable y then $w \equiv x \circ y$ and replacement of $x \circ y$ by the new variable z leads to the gemini equation with less than n quadratic variables.

If t is not a variable, then there is a subterm yz of t where y and z are variables.

If yz is not a twin then there are subterms $s_1, \ldots, s_m$ such that $\langle s_m \rangle_1 = \{y\}$ or $\langle s_m \rangle_1 = \{z\}$, for all $i \quad (i < m)$ both $y \notin \langle s_i \rangle$ and $z \notin \langle s_i \rangle$ and either $\langle yz \circ s_1 \circ \cdots \circ s_m \rangle_1 = \{z\}$ or $\langle yz \circ s_1 \circ \cdots \circ s_m \rangle_1 = \{y\}$. Since $y \in \langle s_m \rangle_1$ or $z \in \langle s_m \rangle_1$, $s_m \equiv v$ but then y and z cannot belong to $\langle yz \circ s_1 \circ \cdots \circ s_m \rangle_1 = \langle uv \rangle_1$.

So yz must be a twin. Consequently $y \circ z$ is a subterm of v. Replacing $y \circ z$ by the new variable leads to equivalent gemini equation with less than n quadratic variables.

(b2) $x \circ t$ is not a twin.
Then there are subterms $s_1, \ldots, s_m$ such that $\langle s_m \rangle_1 = \{x\}$ or $\langle s_m \rangle_1 = \langle t \rangle_1$, for all $i < m \quad x \notin \langle s_i \rangle$ and $\langle s_i \rangle \cap \langle t \rangle = \emptyset$ and such that either $\langle x \circ t \circ s_1 \circ \cdots \circ s_m \rangle_1 = \langle t \rangle_1$ or $\langle x \circ t \circ s_1 \circ \cdots \circ s_m \rangle_1 = \{x\}$.

(b2.1) $\langle s_m \rangle_1 = \{x\}$.
Then $\langle x \circ t \circ s_1 \circ \cdots \circ s_m \rangle_1 = \langle t \rangle_1$ and $s_m \equiv v$. Since $\langle v \rangle_2 = \emptyset$ $s_m \equiv v \equiv x$. But then $x \circ t \circ s_1 \circ \cdots \circ s_{m-1} \equiv u$ and $\langle x \circ t \circ s_1 \circ \cdots \circ s_{m-1} \rangle_1 = \langle x \circ t \rangle_1$ so $m = 1$. Therefore $uv \equiv (x \circ t) \circ x = (t \circ x) \circ x = tx/x = t$ in TS loops.

Replacing uv by t in $w_1 = w_2$, we get equivalent gemini equation with less than n quadratic variables.

(b2.2) $\langle s_m \rangle_1 = \langle t \rangle_1$.
Then $\langle x \circ t \circ s_1 \circ \cdots \circ s_m \rangle_1 = \{x\}$. Since $x \notin \langle t \rangle_1$ and $x \notin \langle s_i \rangle$ (for $i < m$) no s_i is v, so $x \circ t \circ s_1 \circ \cdots \circ s_m$ is a subterm of u. But u has no quadratic variables proving case (b2.2) impossible.

In all casses we reduced the number of quadratic variables to less than n using only identities satisfied in all TS loops. By induction hypothesis all gemini equations with less than n quadratic variables are satisfied in all TS loops. So $w_1 = w_2$ is satisfied by all TS loops as well. □

Here is yet another characterization of gemini equations (compare with [4]).

Theorem 6. *Quadratic equation E is gemini iff there is an equation $I(\cdot, *, /, //, \backslash, \backslash\backslash)$ (in the language $\{\cdot, *, /, //, \backslash, \backslash\backslash\}$), true in all TS loops and such that E is $I(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot)$.*

Proof. A quasigroup is totally symmetric iff all operations $\cdot, *, /, //, \backslash, \backslash\backslash$ coincide. The statement of the Theorem 6 then follows from Theorem 5. □

4. Positive gemini equations

Definition 4. Quadratic functional equation $w_1 = w_2$ is positive if there is no subterm t of either w_1 or w_2 such that $\langle t\rangle_1 = \emptyset$.

The results for positive gemini equations are quite similar to those for general gemini equations and follow readily from them. Theorem 9, however, requires a slight change in the proof.

Theorem 7. *Positive quadratic equation E is gemini iff all $\sim$-classes in $\Gamma(E)$ are small.*

Theorem 8. *Every quasigroup which satisfies a positive quadratic equation which is not a gemini equation is isotopic to a group.*

Theorem 9. *Every positive gemini equation is satisfied by all totally symmetric quasigroups.*

As noted before, the proof of the Theorem 9 is similar to the proof of Theorem 5. The case (a1) is impossible for positive quadratic equations and we should note that in all cases of reduction in the number of quadratic variables, resulting equations are positive.

Theorem 10. *Positive quadratic equation E is (positive) gemini iff there is an equation $I(\cdot, *, /, //, \backslash, \backslash\backslash)$ (in the language $\{\cdot, *, /, //, \backslash, \backslash\backslash\}$) true in all TS quasigroups and such that E is $I(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot)$.*

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