The maximum term and the rank of an entire function.

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Introduction.

We indicate first of all certain notations which we shall use in this work. Let $f(z)=\sum_{0}^{\infty} a_n z^n$ be an entire function of finite order ϱ. Let $\mu(r)$ be its maximum term for $|z|=r$ and $\nu(r)$ its rank; $n(r, a)$ denotes the number of zeros of $f(z)-a$ in $|z| \leqq r$. We write also $n(r, a)=n(r)$ and $M(r)=\operatorname{Max}_{|z|=r} f(z)$.

An entire function $f(z)$ is said to have a as an exceptional value if

$$\limsup_{r\to\infty} \frac{\log n(r, a)}{\log r}=\varrho(a)<\varrho.$$

S. M. SHAH [1] has proved the following theorem.

For a canonical product of integral order $\varrho \geqq 1$ *and genus* $p(=\varrho$ *or* $\varrho-1)$

$$\limsup_{r\to\infty} \frac{n(r)\,\Phi(r)}{\log M(r)}=\infty$$

holds where $\Phi(x)$ *is any positive continuous non-decreasing function of* x *such that*

$$\int_{A}^{\infty} \frac{dx}{x\,\Phi(x)}$$

is convergent.

I prove below some results of the above nature for any entire function.

§ 1.

Theorem 1. (I) *If* $f(z)=\sum_{0}^{\infty} a_n z^n$ *is an entire function of order* ϱ $(0 \leqq \varrho<\infty)$ *and if* $\Phi(x)$ *is a function such that*

$$\log x=o(\Phi(x)),$$

then we have

$$\lim_{r\to\infty} \frac{\nu(r)\,\Phi(r)}{\log M(r)}=\infty.$$

(II) *If the order* ϱ *is such that* $0 < \varrho < \infty$, *then*

$$\limsup_{r\to\infty} \frac{\nu(r)\,\Phi(r)}{\log M(r)} = \infty$$

for any $\Phi(x)$ *tending to infinity.*

We observe that the hypothesis on $\Phi(x)$ in the first part of the above theorem allows the integral

$$\int_A^\infty \frac{dx}{x\,\Phi(x)}$$

to be divergent: for instance by taking $\Phi(x) = \log x \log\log x$.

Proof. (I). It is sufficient to prove that

$$\limsup_{r\to\infty} \frac{\log \mu(r)}{\nu(r)\,\Phi(r)} = 0,$$

as the result then follows since $\log \mu(r) \sim \log M(r)$. Now

$$\log \mu(r) = A + \int_{r_0}^{r} \frac{\nu(x)}{x}\,dx < \nu(r)\left[\log\left(\frac{r}{r_0}\right)\right] + O\ (1).$$

So

$$\log \mu(r) < 2\,\nu(r) \log r.$$

Hence

$$\log \mu(r)/\nu(r)\,\Phi(r) < 2 \log r/\Phi(r)$$

and the result follows from (1).

The proof of (II) is immediate with the help of theorem 1 of S. M. Shah [3].

In the first part of the theorem if

$$\limsup_{r\to\infty} \frac{\log M(r)}{(\log r)^2} < \infty,$$

then $\nu(r)$ can be replaced by $n(r)$ as in such cases $\nu(r) \sim n(r)$ holds [4].

But if we do not impose any restrictions as to the nature of the function, then $\nu(r)$ cannot always be replaced by $n(r)$. For instance if $z = 0$ is an exceptional value of $f(z)$, then

$$\lim_{r\to\infty} \frac{n(r)\,\Phi(r)}{\log M(r)} = 0,$$

because then

$$\log M(r) \sim T r^{\varrho} \qquad (0 < T < \infty)$$

and

$$n(r) = O(r^c) \qquad (c < \varrho).$$

So

$$\frac{n(r)\,\Phi(r)}{\log M(r)} < \frac{K r^c\,\Phi(r)}{\log M(r)} \sim \frac{K'\,\Phi(r)}{r^{\varrho - c}} \to 0$$

where we can take $\Phi(x)$ even as large as x^δ $(0 < \delta < \varrho - c)$.

We are proving now the

Theorem 2. *If a is an exceptional value of $f(z)$ then*

$$\lim_{r\to\infty} \frac{n(r,x)\,\Phi(r)}{\log M(r)} = \infty \quad \textit{for all} \quad x \neq a$$

and for any $\Phi(x)$ tending to infinity.

Proof. Let $\varrho(r)$ be the LINDELÖF approximate order of $f(z)$, then

$$\frac{n(r,a)}{r^{\varrho(r)}} \to 0 \text{ as } r\to\infty,$$

because

$$n(r,a) = O(r^c) \qquad (c<\varrho)$$

and

$$\varrho(r) \to \varrho \text{ as } r\to\infty.$$

Hence the ratio $\dfrac{n(r,x)}{r^{\varrho(r)}}$ has a positive lower bound (see [4], p. 87) for all $x \neq a$.

Now

$$\frac{n(r,x)\,\Phi(r)}{\log M(r)} = \frac{n(r,x)}{r^{\varrho(r)}}\,\frac{r^{\varrho(r)}}{\log M(r)}\,\Phi(r)$$

and further $\log M(r) \leqq r^{\varrho(r)}$ for all $r \geqq r_0$. Hence

$$\frac{n(r,x)\,\Phi(r)}{\log M(r)} \geqq A\,\Phi(r) \to \infty \text{ as } r\to\infty$$

for any $\Phi(r)$ tending to infinity.

Theorem 3. *If $f(z)$ is a canonical product of integral order ϱ and of genus $p=\varrho$ and if $\dfrac{n(r)\,\Phi(r)}{r^{p+1}}$ has a positive lower bound, then*

$$\limsup_{r\to\infty} \frac{n(r)\,\Phi(r)}{\log M(r)} = \infty.$$

Proof.[1]) Since

$$\log M(r) \leqq K\left[r^p \int_a^r \frac{n(x)}{x^{p+1}}\,dx + r^{p+1}\int_r^\infty \frac{n(x)}{x^{p+2}}\,dx\right]$$

it is enough to prove that no finite $C>0$ can satisfy

$$r^p \int_a^r \frac{n(x)}{x^{p+1}}\,dx + r^{p+1}\int_r^\infty \frac{n(x)}{x^{p+2}}\,dx > n(r)\,\Phi(r)\,C \tag{1.1}$$

[1]) We give the proof assuming that the function has no zeros at the origin; if it has, then a slight modification of the proof gives the same result.

for all sufficiently large r, e. g. for $r \geqq R$. Now since $\int^{\infty} \frac{n(x)}{x^{p+2}}$ is always convergent, les us choose R so that

$$\int_R^{\infty} \frac{n(x)}{x^{p+2}} < \varepsilon.$$

Let us suppose that (1. 1) holds for $r = R$, then

$$R^p \int_a^R \frac{n(x)}{x^{p+1}} dx + R^{p+1} \int_R^{\infty} \frac{n(x)}{x^{p+2}} dx > n(R) \Phi(R) C, \tag{1. 2}$$

and

$$n(x) = O(x^{p+\varepsilon}).$$

So

$$\int_a^R \frac{n(x)\, dx}{x^{p+1}} < k \int_a^R x^{\varepsilon-1}\, dx = o(R).$$

Hence the left hand side of (1. 2) $= o(R^{p+1})$. So

$$\frac{n(R)\, \Phi(R)}{R^{p+1}} \to 0$$

in contradiction with our hypothesis, and this completes the proof of the theorem.

§ 2.

T. VIJAYA RAGHAVAN has proved [5] that

$$M'(r) \geqq \frac{M(r)}{r} \frac{\log M(r)}{\log r} \qquad \text{for} \quad r > r_0(f).$$

We prove here an analogous result for $\mu(r)$:

Theorem 4.

$$\mu'(r) \geqq \frac{\mu(r)}{r} \frac{\log \mu(r)}{\log r}$$

is valid for every $r > r_0(f)$.

Proof.

$$\log \mu(r) = A + \int^r \frac{\nu(x)}{x} dx\,;$$

hence

$$\frac{\mu'(r)}{\mu(r)} = \frac{\nu(r)}{r} \tag{2. 1}$$

and

$$\log \mu(r = \log |a_n| + n \log r \quad \text{for} \quad R_n \leqq r < R_{n+1}. \tag{2. 2}$$

Since for any entire function

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$$

holds, we have

(2. 3) $$|a_n| < 1 \quad \text{for} \quad n \geqq n_0.$$

Hence log $|a_n|$ is negative and from (2. 2) we get

(2. 4) $$\frac{\log \mu(r)}{\log r} \leqq \nu(r) \text{ for } r \geqq r_0(f).$$

Combining (2. 1) and (2. 4) we get the result of the theorem.

§ 3.

Theorem 5. *If* $\mu(r_0) > 1$ *and one of the integrals*

$$I_1 = \int_{r_0}^{\infty} \frac{\log \mu(t)\, dt}{t^{m+1}}$$

and

$$I_2 = \int_{r_0}^{\infty} \frac{\nu(t)}{t^{m+1}}\, dt$$

converges resp. diverges, then the other converges resp. diverges too.

Proof.

$$\int_{r_0}^{r} \frac{\nu(t)}{t}\, dt = \log \mu(r) - \log \mu(r_0).$$

Hence

$$\int_{r_0}^{u} \frac{dr}{r^{m+1}} \int_{r_0}^{r} \frac{\nu(t)}{t}\, dt = \int_{r_0}^{u} [\log \mu(r) - \log \mu(r_0)]\, \frac{dr}{r^{m+1}} =$$

$$= \left[\frac{\log \mu(r) - \log \mu(r_0)}{-m r^m}\right]_{r_0}^{u} + \frac{1}{m} \int_{r_0}^{u} \frac{\nu(r)\, dr}{r^{m+1}} =$$

$$= \frac{\log \mu(u) - \log \mu(r_0)}{-m u^m} + \frac{1}{m} \int_{r_0}^{u} \frac{\nu(r)\, dr}{r^{m+1}} =$$

$$= \int_{r_0}^{u} [\log \mu(r) - \log \mu(r_0)]\, \frac{dr}{r^{m+1}} =$$

$$= \int_{r_0}^{u} \frac{\log \mu(r)\, dr}{r^{m+1}} - \log \mu(r_0) \int_{r_0}^{u} \frac{dr}{r^{m+1}} = \int_{r^0}^{u} \frac{\log \mu(r)\, dr}{r^{m+1}} + \frac{\log \mu(r_0)}{m} \left[\frac{1}{u^m} - \frac{1}{r_0^m}\right].$$

Thus

$$\int_{r_0}^{r} \frac{\log \mu(t)}{t^{m+1}}\, dt + \frac{\log \mu(r_0)}{m}\left[\frac{1}{r^m} - \frac{1}{r_0^m}\right] = \frac{\log \mu(r) - \log \mu(r_0)}{-m r^m} + \frac{1}{m}\int_{r_0}^{r} \frac{\nu(t)\, dt}{t^{m+1}}$$

and so

$$m \int_{r_0}^{r} \frac{\log \mu(t)\, dt}{t^{m+1}} - \frac{\log \mu(r_0)}{r_0^m} + \frac{\log \mu(r)}{r^m} = \int_{r_0}^{r} \frac{\nu(t)}{t^{m+1}}\, dt. \tag{3. 1}$$

We suppose now that I_1 is convergent; then

$$\varepsilon > \int_{r}^{2r} \frac{\log \mu(t)\, dt}{t^{m+1}} > \frac{\log \mu(r)}{m r^m}\left[1 - \frac{1}{2^m}\right] \text{ for every } \varepsilon > 0.$$

Consequently

$$\frac{\log \mu(r)}{r^m} \to 0.$$

Hence

$$m I_1 + K = I_2 \qquad \left(K = -\frac{\log \mu(r_0)}{r_0}\right)$$

holds and this implies the convergence of I_2.

Similarly if I_2 is convergent, then

$$m \int_{r_0}^{r} \frac{\log \mu(t)\, dt}{t^{m+1}} + \frac{\log \mu(r)}{r^m} < K'$$

and as

$$\int_{r_0}^{r} \frac{\log \mu(t)\, dt}{t^{m+1}} > \log \mu(r_0)\, \frac{1}{m}\left[\frac{1}{r_0^m} - \frac{1}{r^m}\right] > 0$$

both terms on the left hand side are positive and this secures the convergence of I_1.

From the convergence of $\int_{r_0}^{\infty} \frac{\nu(t)}{t^{m+1}}\, dt$ we can also deduce that $\log \mu(r) = o(r^m)$.

Further we get from (3. 1) that

$$\int_{r_0}^{r} \frac{\nu(t)\, dt}{t^{m+1}} > \int_{r_0}^{r} \frac{\log \mu(t)\, dt}{t^{m+1}}.$$

Hence the divergence of I_2 follows from the divergence of I_1.

Now let I_2 be divergent. Then I_1 will also be divergent for if it were convergent, then by the results established above I_2 will also be convergent in contradiction with our hypothesis.

§ 4.

In [6] I have proved that for $0 < r < R$

$$\frac{M(R)}{m(r)} \geqq \left(\frac{R}{r}\right)^{n(r)}$$

where

$$m(r) = \operatorname*{Min}_{|z|=r} |f(z)|$$

and as usual

$$M(R) = \operatorname*{Max}_{|z|=R} |f(z)|.$$

Here we shall prove further

Theorem 6. *If $f(z)$ is an entire function having no zeros in the unit circle, then*

$$\frac{M(R)}{m(r)} \geqq \left(\frac{R}{r}\right)^{\frac{N(R)}{\log R}} \qquad (0 < r < R)$$

where

$$N(R) = \int_0^r \frac{n(t)}{t}\, dt.$$

Proof.

$$N(R) - N(r) = \int_r^R \frac{n(t)}{t}\, dt = \int_0^R \frac{n(t)}{t}\, dt - \int_0^r \frac{n(t)}{t}\, dt \leqq \tag{4. 1}$$

$$\leqq \log M(R) - \log m(r)$$

by JENSEN's theorem. $N(x)$ is an increasing convex function of $\log x$. If we draw the graph of the function $N(x)$, it will pass through the origin. Let O be the origin and $A\,(\log R, N(R))$, $B\,(\log r, N(r))$, be two points on the graph. Then the slope of OA is greater than the slope of OB. Hence

$$\frac{N(R)}{\log R} \geqq \frac{N(r)}{\log r}$$

and it follows that

$$\frac{N(R) - N(r)}{\log R - \log r} \geqq \frac{N(R)}{\log R}.$$

Thus (4. 1) gives

$$\frac{N(R)}{\log R} \leqq \frac{\log M(R) - \log m(r)}{\log R - \log r}$$

from which the result of the theorem follows.

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References.

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