Generalized complemented and quasicomplemented lattices.

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§ 1. Introduction.

It is known that complemented lattices [1]) play a very important role both in abstract lattice theory and in its applications to various branches of mathematical researches.

In this paper we shall be concerned with a generalization of the concept of complemented lattices [2]). First we recall the well-known definitions concerning the classic concept of complementation ([B], p. 23):

Definition 1. *By a complement of an element x of a lattice L with O and I is meant an element $y \in L$ such that $x \cap y = O$ and $x \cup y = I$; L is called complemented if all its elements have complements.*

Definition 2. *A lattice L is called relatively complemented if all its closed intervals* [3]) *$[a, b]$ are complemented. By a "relative complement of $x (\in [a, b])$ in the interval $[a, b]$" is meant an element $y \in [a, b]$ such that $x \cap y = a$ and $x \cup y = b$.*

In connection with the subject of our paper we should like to call the reader's attention to the following points of view which seem to motivate our treatment below:

1°. The presence of the elements O, I is, in view of lattices of infinite length, a too special requirement. It seems therefore to be useful *to give a generalization of the concept of complemented lattices which is, in the case of lattices with O, I, equivalent to the classic definition, but may be applied also for lattices without O, I.*

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[1]) For the usual notations and terminology see G. BIRKHOFF: Lattice theory, *Amer. Math. Soc. Colloquium Publications*, vol. **25**, revised edition, New York, 1948. — To this work we refer in the following by [B].

[2]) In a quite different direction the author has given a generalization of the notion of complement; see G. SZÁSZ, Dense and semicomplemented lattices, *Nieuw Archief voor Wiskunde*, (3), vol. **1** (1953), 42—44.

[3]) By a closed interval $[a, b]$ (where $a \leqq b$) is meant the set of all elements x of L for which $a \leqq x \leqq b$; $[a, b]$ is a sublattice of L.

2°. The definitions above enable us to define, also for lattices without O and I, relative complements, but complements not. This means that relative complementedness does not imply, in general, complementedness, although the assumption to be relatively complemented is a more stronger requirement for the structure of the lattice, than complementedness. It seems therefore to be desirable *to extend the notion of complementedness in such a manner that relative complementedness shall always imply "complementedness in the generalized sense"*.

Now, in section 2 we give a generalized definition of complemented lattices satisfying the requirements given in 1°—2°. Afterwards we show, in section 3, that "complemented lattices in the generalized sense" have similar properties as "complemented lattices in the classic sense". Finally, in section 4 we introduce, as a further generalization, "quasicomplemented lattices" as lattices all of whose intervals $[a, b]$ with $a \neq O, b \neq I$ are complemented in the sense of section 2.

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§ 2. Complemented lattices in the generalized sense.

In view of the preceding considerations we give the following generalized definition of complemented lattices:

Definition 3. *A lattice L (either having elements O, I or not) is called complemented (in the generalized sense), if given an arbitrary pair of elements u, v, for every element $a (\in L)$ there exists at least one element x in L such that $a \cap x \leqq u$, $a \cup x \geqq v$.*

For the element x of definition 3 we introduce the following symbol and terminology:

Definition 4. *Let a, u, v be arbitrary elements of the lattice L. If there exists an element x such that $a \cap x \leqq u, a \cup x \geqq v$, then it will be called a "(u, v)-complement of a in L" and will be denoted by a_u^v.*

This means, that

$$a \cap a_u^v \leqq u, \quad a \cup a_u^v \geqq v$$

hold for arbitrary elements a, u, v of L (if any a_u^v exists at all). Thus our definition 3 may be enounced also in the following form: *A lattice L is called complemented if all elements a $(\in L)$ have (u, v)-complements for an arbitrary pair of elements u, v.*

It follows immediately from the definition 4 that (u, v)-complements are a fortiori (t, w)-complements for all $t > u$, $w < v$. Moreover, an (O, I)-complement of an element a is a complement a' of a in the usual sense.

The complement a', defined by definition 1, has the property that conversely, a is a complement of a', i. e. the property of being complement is symmetric. It is obvious from definition 4 that the property of being a (u, v)-complement is also symmetric: for all a, a is an (u, v)-complement of a_u^v.

After these preliminary remarks we show

Theorem 1. *For lattices with O and I, definition 3 and definition 1 are equivalent.*

For, by taking $a_u^v = a'$ for all a of a lattice L, complemented in the sense of definition 1, there follows the existence of a_u^v for arbitrary u, v, so that L is complemented also in the sense of definition 3. Conversely, let L be complemented in the sense of definition 3. If it has elements O and I, then there exists at least one $a_O^I = a'$ for all a in L, so that L is complemented also in the sense of definition 1.

Owing to this equivalence, it is unnecessary to make distinction between the concepts "complemented lattices in the sense of definition 1" and "complemented lattices in the sense of definition 3". Consequently, in the following we may regard, as definition of complemented lattices, always the (more general) definition 3.

Concerning relatively complemented lattices we show

Theorem 2. *Any relatively complemented lattice L is complemented.*

For, let a, u, v be arbitrary elements of L. Then

$$a \cap u \leqq a \leqq a \cup v.$$

L being relatively complemented, there exists a relative complement t of a in the interval $[a \cap u, a \cup v]$. Therefore we have

$$a \cap t = a \cap u, \qquad a \cup t = a \cup v,$$

i. e.

$$a \cap t \leqq u, \qquad a \cup t \geqq v,$$

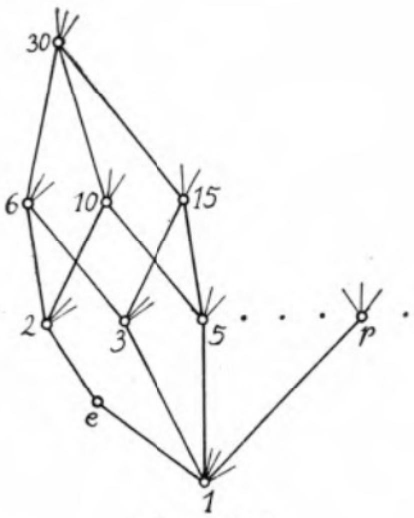
completing the proof of our theorem.

Theorems 1 and 2 show that definition 3 gives in fact a generalization of complemented lattices, suitable for our points of view 1°—2° in the introduction.

We give now two examples for complemented lattices in which at least one of the elements O, I does not exist:

Example 1. Consider the set R of all positive integers $k, l, m, \ldots$ with square-free prime factorization. Let $k \leqq l$ mean that k divides l, then R becomes a lattices with the O-element 1; however, R has no I-element. One sees easily that k_l^m exists for all k, l, m in R.

Example 2. Consider now the lattice L whose diagram differs from that of R in the preceding example only in that the interval $[1, 2]$ of L contains a new element e with the property $1 < e < 2$:



Example 2

It is easy to see that $e \cap k \leqq 2 \cap k, e \cup k = 2 \cup k$ for all $k \in L$. These relations enable us to take for e_k^l always 2_k^l (for all k, $l \in L$). Hence we have an example for a non-modular complemented lattice without I.

We give an example for a complemented lattice without O and I.

Example 3. Let us consider a line g and assign a point P on g. Let S mean the collection of all sets[4]) $\mathfrak{A}, \mathfrak{B}, \ldots$ of g which satisfy the following conditions: (i) every element $\mathfrak{A}$ of S consists of a finite number of left closed, right open intervals $a_1, a_2, \ldots, a_r$ of finite length >0; (ii) one of $a_i (i \leqq r)$ contains the assigned point P in its interior. Since the set-theoretical union and intersection of such sets belong obviously to the same type, S can be made a lattice with $\mathfrak{A} \cap \mathfrak{B} = \mathfrak{A} \mathbin{\pmb{\cap}} \mathfrak{B}$ and $\mathfrak{A} \cup \mathfrak{B} = \mathfrak{A} \mathbin{\pmb{\cup}} \mathfrak{B}$.[5]) Moreover, S is a distributive lattice and it has neither O- nor I-element. We show that S is complemented.

For, let $\mathfrak{A}, \mathfrak{U}, \mathfrak{B}$ be arbitrary elements of S. Let $\mathfrak{O}$ denote the void set (which does not belong to S); further, let $\mathfrak{Z}$ be an element of S such that $\mathfrak{Z} \leqq \mathfrak{U}$. We prove that the element $\mathfrak{X} = (\mathfrak{B} - \mathfrak{A}) \mathbin{\pmb{\cup}} \mathfrak{Z}$ is a $(\mathfrak{U}, \mathfrak{B})$-complement of $\mathfrak{A}$[6]). Indeed, using also the distributivity, we have

$$\mathfrak{A} \cup \mathfrak{X} = \mathfrak{A} \mathbin{\pmb{\cup}} \mathfrak{X} = \mathfrak{A} \mathbin{\pmb{\cup}} (\mathfrak{B} - \mathfrak{A}) \mathbin{\pmb{\cup}} \mathfrak{Z} = \mathfrak{B} \mathbin{\pmb{\cup}} \mathfrak{Z} \mathbin{\pmb{\supset}} \mathfrak{B}$$

and

$$\mathfrak{A} \cap \mathfrak{X} = \mathfrak{A} \mathbin{\pmb{\cap}} [(\mathfrak{B} - \mathfrak{A}) \mathbin{\pmb{\cup}} \mathfrak{Z}] = [\mathfrak{A} \mathbin{\pmb{\cap}} (\mathfrak{B} - \mathfrak{A})] \mathbin{\pmb{\cup}} (\mathfrak{A} \mathbin{\pmb{\cap}} \mathfrak{Z}) = \mathfrak{O} \mathbin{\pmb{\cup}} (\mathfrak{A} \mathbin{\pmb{\cap}} \mathfrak{Z}) \mathbin{\pmb{\subset}} \mathfrak{Z} \mathbin{\pmb{\subset}} \mathfrak{U},$$

completing the proof of the complementedness of S.

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[4]) By $\pmb{\subset}$, $\pmb{\cup}$ and $\pmb{\cap}$ we denote inclusion, union and intersection, respectively, in the set-theoretical sense. By $\mathfrak{A} - \mathfrak{B}$ we mean the set consisting of all elements of $\mathfrak{A}$ which are not included in $\mathfrak{B}$. Finally, $\{a, b, \ldots\}$ is the set consisting of the elements $a, b, \ldots$.

[5]) Consequently, $\mathfrak{A} \leqq \mathfrak{B}$ means $\mathfrak{A} \mathbin{\pmb{\subset}} \mathfrak{B}$.

[6]) We call the attention of the reader to that the set $\mathfrak{B} - \mathfrak{A}$ does not belong to S. That is why it was necessary to take $(\mathfrak{B} - \mathfrak{A}) \mathbin{\pmb{\cup}} \mathfrak{Z}$ for a $(\mathfrak{U}, \mathfrak{B})$-complement of $\mathfrak{A}$.

§ 3. Complemented modular and distributive lattices.

In this section we shall give generalizations of some well-known theorems of the theory of complemented modular and distributive lattices having elements O and I.

We recall the following results (i)—(iii):

(i) Any complemented modular lattice with O and I is relatively complemented ([B], p. 114, theorem 1).

(ii) In any complemented distributive lattice with O and I, we have $(a \cup b)' = a' \cap b'$ and $(a \cap b)' = a' \cup b'$ for all a, b ([B], p. 152, theorem 1).

(iii) Let a, b be arbitrary elements of a complemented modular lattice L with O, such that $a \leqq b$. If x is a relative complement of the element a in the interval $[O, b]$, then it is a complement of $a \cup b'$ [7]).

We show, by theorems 3—5, that the "natural generalizations" of these theorems hold also for complemented lattices which have not necessarily elements O and I.

Theorem 3. *Any complemented modular lattice L (with or without elements O and I) is relatively complemented.*

Proof. Let a, b and x be arbitrary elements of L such that

(1) $$(O \leqq)\, a \leqq x \leqq b\, (\leqq I).$$

L being complemented, x_a^b exists. Consider now, similarly to the usual proof of (i), an element

(2) $$t = (a \cup x_a^b) \cap b = a \cup (x_a^b \cap b).$$

Obviously $a \leqq t \leqq b$; hence

(3) $$a \leqq x \cap t \leqq x \leqq x \cup t \leqq b.$$

On the other hand, by our assumptions (2), (1) and by modularity, we have

(4. 1) $$x \cap t = x \cap b \cap (a \cup x_a^b) = x \cap (a \cup x_a^b) = a \cup (x_a^b \cap x) \leqq a \cup a = a,$$

(4. 2) $$x \cup t = x \cup a \cup (x_a^b \cap b) = x \cup (x_a^b \cap b) = (x \cup x_a^b) \cap b \geqq b \cap b = b.$$

Relations (3), (4. 1) and (4. 2) imply $x \cap t = a, x \cup t = b$; i. e. t is a relative complement of x in $[a, b]$.

By theorems 2 and 3 we have the following

Corollary. *In the case of modular lattices complementedness and relative complementedness are equivalent concepts.*

However, as (besides simple classic counterexamples of finite length, also) example 2 shows, this equivalence does not remain valid for non-modular lattices: in fact, the interval $[1, 2]$ forms a non-complemented sublattice of L.

[7]) Statement (iii) is given as an exercise without proof in [B], p. 115, ex. 2. It may be verified on the same lines as the statement (α) of our theorem 6.

Theorem 4. *In any complemented distributive lattice L, $a^r_u \cap b^r_u$ may be taken as $(a \cup b)^r_u$ and $a^r_u \cup b^r_u$ as $(a \cap b)^r_u$ for arbitrary elements a, b, u, v of L.*

Proof. By distributivity,

$$(a \cup b) \cup (a^r_u \cap b^r_u) = (a \cup b \cup a^r_u) \cap (a \cup b \cup b^r_u) \geqq v \cap v = v,$$

$$(a \cup b) \cap (a^r_u \cap b^r_u) = (a \cap a^r_u \cap b^r_u) \cup (b \cap a^r_u \cap b^r_u) \leqq u \cup u = u;$$

and dually for $a \cap b$.

This analogy to the usual complements in Boolean algebras enables us to make the following generalization: *By a Boolean algebra is meant a complemented distributive lattice (whether it has elements O, I or not).*

Statement (iii) may be generalized as follows:

Theorem 5. *Let a, b and u, v be arbitrary elements of the complemented modular lattice L (with or without elements O, I) such that $u \leqq a \leqq b$. If x is a relative complement of the element a in the interval $[u, b]$, then it is a (u, v)-complement of $a \cup b^r_u$.*

Proof. By our assumptions we obtain

$$a \cup x = b, \quad a \cap x = u. \tag{5}$$

Hence

$$(a \cup b^r_u) \cup x = (a \cup x) \cup b^r_u = b \cup b^r_u \geqq v. \tag{6}$$

On the other hand, (5) implies obviously the inequality $x \leqq b$, $u \leqq x$. Hence we have, by making use also of modularity for the elements $a \leqq b$,

$$(a \cup b^r_u) \cap x \leqq (a \cup b^r_u) \cap b \leqq a \cup (b^r_u \cap b) \leqq a \cup u. \tag{7}$$

By (7) and the obvious inequality

$$(a \cup b^r_u) \cap x \leqq x$$

we get, using again the modularity for the elements $u \leqq x$,

$$(a \cup b^r_u) \cap x \leqq (a \cup u) \cap x = u \cup (a \cap x) = u \cup u = u, \tag{8}$$

completing the proof.

If L has O-element and the element x (of the preceding theorem) is a relative complement of a not in $[u, b]$, but in $[O, b]$, then we may prove the following stronger assertion:

Theorem 6. *Let a, b and v be arbitrary elements of a complemented modular lattice L with O such that $a \leqq b$. Let further x be a relative complement of a in the interval $[O, b]$.*

(α:) *Then x is an $[O, v]$-complement of $a \cup b^r_u$ for all $u \leqq a$;*

(β:) *Moreover, for distributive L, x is a (u, v)-complement of $a \cup b^r_u$ for all u in L.* (For $u \leqq a$, according to the assertion (α), x is also an (O, v)-complement of $a \cup b^r_u$.)

Proof. First we prove (α). By our assumptions we obtain

$$a \cup x = b, \quad a \cap x = O. \tag{9}$$

Hence we have, just as in the proof of the preceding theorem,

$$(a \cup b_u^v) \cup x \geqq v, \tag{10}$$

and $(a \cup b_u^v) \cap x \leqq a \cup u$. From the latter, by $u \leqq a$, we get

$$(a \cup b_u^v) \cap x \leqq a.$$

This inequality, together with the obvious relation $(a \cup b_u^v) \cap x \leqq x$ and (9), implies

$$(a \cup b_u^v) \cap x \leqq a \cap x = O. \tag{11}$$

Inequalities (10) and (11) yield our assertion (α).

As for the assertion (β), inequality (10) may be found again in the same manner as (6). Further, by distributivity, using also (9), we have

$$(a \cup b_u^v) \cap x = (a \cap x) \cup (b_u^v \cap x) = b_u^v \cap x \leqq b_u^v \cap b \leqq u$$

for arbitrary u, v, completing so the proof.

§ 4. Quasicomplemented lattices.

Consider now the following

Example 4. Let L denote the set of all finite subsets $\mathfrak{A}, \mathfrak{B}, \ldots$ of any countable set $\mathfrak{H}$ (including also the void set $\mathfrak{O}$). Defining in L the operations $\cap$ and $\cup$ as in example 3 we make it into a distributive lattice with the O-element $\mathfrak{O}$. It is easy to prove that L is relatively complemented. Now if we adjoin to L, as element I, the whole set $\mathfrak{H}$, then we give rise to a new lattice L^* with O and I. But, one sees easily that L^* is no longer relatively complemented (not even complemented). However, every interval $[a, b]$, $b \neq I$, of L^* is obviously complemented.

For lattices of such type we introduce the following

Definition 5. *A lattice L is called quasicomplemented if all its elements $a \neq O$, I (if O, I exist at all) have (u, v)-complements for arbitrary pairs of elements $u \neq O$, $v \neq I$.*

Complemented lattices are a fortiori quasicomplemented. Conversely, *quasicomplemented lattices without O and I are also complemented.*

Clearly, the statements of theorems 3—6 hold essentially for quasicomplemented lattices, excluded only the eventual intervals bounded by O or I. In fact, instead of theorem 3 we may prove, by the same arguments as above, the following

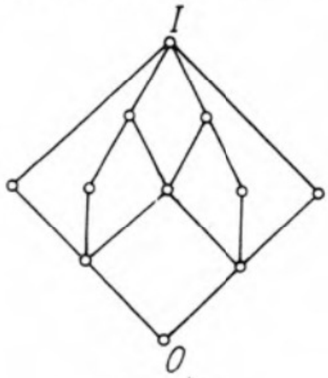
Theorem 3'. *Every interval $[a, b]$ of a quasicomplemented modular lattice is complemented provided that neither $a = O$ nor $b = I$.*

Further, all statements of theorems 4—6 hold also for quasicomplemented lattices with the obvious restrictions

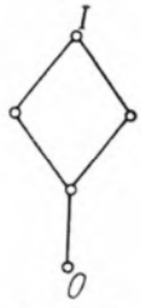
$$a \neq O, I; \quad b \neq O, I; \quad u \neq O; \quad v \neq I;$$

moreover, statement (β) of theorem 6 is for $a = O$ trivial provided that $b \neq O, I$; $u \neq O$; $v \neq I$.

For quasicomplemented lattices of finite length, which are however not complemented, we give the following two examples:



Example 5



Example 6

As an example for a quasicomplemented, but non-complemented lattice of infinite length we refer to our example 4 above. Examples 4—5 demonstrate the following obvious statement. Let L be a complemented lattice with or without O, I. Let us define two new elements $\bar{O}, \bar{I}$ such that $\bar{O} \leqq x, \bar{I} \geqq x$ for all $x \in L$. If we adjoin one or both elements $\bar{O}, \bar{I}$ to L, then we get a new lattice $\bar{L}$ which is in general already not complemented, but quasicomplemented.

However, as example 6 shows, not every quasicomplemented lattice may be constructed by this method.

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