On the functional equation of autodistributivity.

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A function of two variables $M(x, y)$ is called autodistributive (on the right side) if

$$(1) \qquad M[M(x, y), z] = M[M(x, z), M(y, z)].$$

The functional equation (1) resp. a similar equation was first examined by C. RYLL-NARDZEWSKI [3].[1]) He proved that all strictly monotonic, continuous and symmetric[2]) solutions of (1) are quasiarithmetic means i. e. they have the form

$$(2) \qquad M(x, y) = f^{-1}\left[\frac{1}{2}f(x) + \frac{1}{2}f(x)\right],$$

where $f(t)$ is an arbitrary strictly monotonic, continuous function and $f^{-1}(t)$ is its inverse function.

On the other hand B. KNASTER [2] has proved that all strictly monotonic, continuous, autodistributive and symmetric functions are bisymmetric[3]) and hence they all have the form (2).

J. ACZÉL[4]) examined first the functional equation (1) without supposing symmetry. He proved that all strictly monotonic and twice differentiable solutions of (1) are quasilinear means i. e. they have the form

$$(3) \qquad M(x, y) = f^{-1}[pf(x) + qf(y)], \qquad (p + q = 1).$$

The object of the present paper is to show that if autodistributivity *on both sides* is required, then it is sufficient to suppose the differentiability *in first order* for the validity of (3).

We prove the

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[1]) The numbers in brackets refer to the Bibliography at the end of this paper.

[2]) $M(x, y)$ is called symmetric if $M(x, y) = M(y, x)$. C. RYLL-NARDZEWSKI has involved the symmetry of $M(x, y)$ in the functional equation

$$M[x, M(y, z)] = M[M(x, y), M(z, x)]$$

which implies also the functional equation (1). See [3].

[3]) A function $M(x, y)$ is called bisymmetric if

$$M[M(x, y), M(u, v)] = M[M(x, u), M(y, v)].$$

The functional equation of bisymmetry was solved by J. ACZÉL supposing strict monotony and continuity. See [1] and [2[.

[4]) Oral communication. ACZÉL conjectured also that the condition of differentiability in second order might be omitted.

Theorem. *If the function $M(x, y)$ strictly monotonic in both variables and (once) derivable, satisfies the functional equations*

(1a) $$M[M(x,y),z]=M[M(x,z),M(y,z)],$$

(1b) $$M[z,M(x,y)]=M[M(z,x),M(z,y)]$$

then and only then $M(x,y)$ has the form

(3) $$M(x,y)=f^{-1}[pf(x)+qf(y)], \qquad (p+q=1)$$

where $f(t)$ is an arbitrary strictly monotonic and differentiable function, the inverse function of wich is $f^{-1}(t)$ and p is a constant.

Proof. We shall make use of two lemmas.

Lemma I. If the strictly monotonic and derivable function $M(x,y)$ satisfies e. g. the functional equation (1a) *then*

(4) $$M(t,t)=t$$

and

$$M_1(t,t)=p, \quad M_2(t,t)=q$$

are constants with

$$p+q=1$$

where the indices $_1$ resp. $_2$ denote the partial differential quotients of $M(x,y)$ with respect to the first resp. second variable.

Proof of Lemma I.[5]) First we remark that $M(x,y)$ is reflexive

(4) $$M(t,t)=t$$

as putting $x=y=z$ in (1a) we have

$$M[M(x,x),x]=M[M(x,x),M(x,x)]$$

which by the monotony of $M(x,y)$ implies (4).

If we take now the derivative of (1a) with respect to x then we get

$$M_1[M(x,y),z]\,M_1(x,y)=M_1[M(x,z),M(y,z)]\,M_1(x,z).$$

Let us put here $y=x$ then by (4)

$$M_1(x,z)\,M_1(x,x)=M_1[M(x,z),M(x,z)]\,M_1(x,z),$$

consequently

(5) $$M_1(x,x)=M_1[M(x,z),M(x,z)]=p \text{ (constant).}$$

Further by differentiating (4) we have

(6) $$M_1(t,t)+M_2(t,t)=1.$$

(4), (5) and (6) contain the assertion of *Lemma I.*

Lemma II. If the strictly monotonic and continuous function $M(x,y)$ satisfies the functional equations (1a) *and* (1b) *then the functions*

$$t_1=M(x,u),$$
$$t_2=M(x,v),$$
$$t_3=M(y,u),$$
$$t_4=M(y,v)$$

[5]) This Lemma I is due to J. Aczél [4]).

— *all considered as functions of the four variables* x, y, u, v — *are not independent.*

Proof of Lemma II. We prove Lemma II indirectly by supposing the independence of the functions t_1, t_2, t_3, t_4. Then $M(x, y)$ is bisymmetric

$$M[M(t_1, t_2), M(t_3, t_4)] = M[M(t_1, t_3), M(t_2, t_4)]$$

since by (1a) and (1b)

$$M[M(x, y), M(u, v)] = M\{M[x, M(u, v)], M[y, M(u, v)]\} = \\ = M\{M[M(x, u), M(x, v)], M[M(y, u), M(y, v)]\}$$

and

$$M[M(x, y), M(u, v)] = M\{M[M(x, y), u], M[M(x, y), v]\} = \\ = M\{M[M(x,u), M(y,u)], M[M(x,v), M(y,v)]\}.$$

Hence[6]) $M(x, y)$ is a quasilinear function

$$M(x, y) = f^{-1}[a f(x) + b f(y) + c].$$

But then

$$f(t_1) = a f(x) + b f(u) + c,$$
$$f(t_2) = a f(x) + b f(v) + c,$$
$$f(t_3) = a f(y) + b f(u) + c,$$
$$f(t_4) = a f(y) + b f(v) + c,$$
$$f(t_1) + f(t_4) = f(t_2) + f(t_3) = a[f(x) + f(y)] + b[f(u) + f(v)] + 2c$$

in contradiction with the supposed independence of t_1, t_2, t_3, t_4.

Now we can turn our attention to the

Proof of the Theorem. We proved in Lemma II. that the functions $M(x, u), M(x, v), M(y, u), M(y, v)$ are not independent hence their functional determinant must vanish:

$$\begin{vmatrix} M_1(x, u), & 0, & M_2(x, u), & 0 \\ M_1(x, v), & 0, & 0, & M_2(x, v) \\ 0, & M_1(y, u), & M_2(y, u), & 0 \\ 0, & M_1(y, v), & 0, & M_2(y, v) \end{vmatrix} = 0$$

or what is the same

$$M_1(x, u)\, M_2(x, v)\, M_2(y, u)\, M_1(y, v) = \\ = M_2(x, u)\, M_1(x, v)\, M_1(y, u)\, M_2(y, v).$$

If we put $u = y$ then taking Lemma I into account we get

$$\frac{M_1(x, y)}{M_2(x, y)} = \frac{p}{q}\, \frac{M_1(x, v)}{M_2(x, v)}\, \frac{M_2(y, v)}{M_1(y, v)}.$$

[6]) See [1]. Moreover here (4) implies $c = 0$ and $a + b = 1$.

Let us keep $v = v_0$ constant and define the function $f(t)$ by the equation

$$f'(t) = \frac{M_1(t, v_0)}{M_2(t, v_0)}$$

so we arrive to

$$\frac{M_1(x, y)}{M_2(x, y)} = \frac{p f'(x)}{q f'(y)}.$$

This can be written in the form

$$\begin{vmatrix} M_1(x, y) & M_2(x, y) \\ p f'(x) & q f'(y) \end{vmatrix} = 0$$

which shows the dependence of the functions $M(x, y)$ and $pf(x) + qf(y)$ i. e. there exists a function $\varphi(t)$ such that

$$\varphi[M(x, y)] = p f(x) + q f(y), \quad (p + q = 1).$$

Putting $y = x$ and taking (4) into account this gives

$$\varphi(x) = (p+q) f(x) = f(x).$$

Hence the solution is

$$M(x, y) = f^{-1}[p f(x) + q(y)], \qquad (p + q = 1)$$

Finally we verify that this solution really satisfies the functional equation (1a) and (1b):

$$\begin{aligned} M[M(x, y), z] &= f^{-1}\{p[pf(x) + qf(y)] + qf(z)\}, \\ M[M(x, z), M(y, z)] &= f^{-1}\{p[pf(x) + qf(z)] + q[pf(y) + qf(z)]\} = \\ &= f^{-1}\{p[pf(x) + qf(y)] + (p+q) q f(z)\} \end{aligned}$$

so (1a) is satisfied and similarly also (1b). This completes the proof of our Theorem.

Remark. Looking back to our proof we see that we have proved our assertion (3) in both cases: if the system of functions in Lemma II is not independent and if it is independent. But in the latter case the formula (3) thus proved offers on the other hand a contradiction with the supposed independence.

Bibliography.

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[2] B. Knaster, Sur une équivalence pour les fonctions, *Colloquium Mathematicum*, **2** (1949), 1—4.
[3] C. Ryll-Nardzewski, Sur les moyennes, *Studia Mathematica*, **11** (1949), 31—37.

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