The abelian groups with torsion-free endomorphism ring.

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Consider an additively written abelian group G. We say that G is a torsion group if every element in G is of finite order; moreover, G is torsion-free if every element $\neq 0$ in G is of infinite order. If G is neither torsion-free nor a torsion group, then G is called a mixed group. We call also the endomorphism ring $E(G)$ of G torsion-free if the additive group of the ring $E(G)$ is torsion-free. Clearly $E(G)$ is torsion-free if G is a torsion-free group. There exist, however, non-torsion-free groups the endomorphism ring of which is torsion-free. In what follows we determine all abelian groups having this property.

We denote by x, a, b, c elements of groups. The other small Latin letters serve to denote rational integers. In particular p will denote always a prime number. By an *algebraically closed abelian group* we mean an abelian group G in which every equation $nx = a$ admits a solution $x \in G$ for any $a \in G$ and any natural number n. Clearly the group G is algebraically closed if and only if $nG = G$ holds for $n = 1, 2, 3, \ldots$. It is well-known that an algebraically closed abelian group is the direct sum of groups isomorphic to R and $C(p^\infty)$ where R denotes the additive group of all rational numbers and $C(p^\infty)$ denotes PRÜFER's group of type (p^∞), i. e., the additive group mod 1 of all rational numbers with p-power denominators. [2].[1]) We denote by $C(p)$ a group of order p. If for an abelian group G the equation $pG = G$ holds, then we say that G is *closed for the prime p*. If G contains an element of order p, then we call p an actual prime for the group G.

Now we can formulate the following

Theorem. *The endomorphism ring $E(G)$ of an abelian group G is torsion-free if and only if G can be represented in the form of the direct sum:*

$$G = A + B \tag{1}$$

where A is an algebraically closed torsion group, and B is a torsion-free group which is closed for each actual prime p for A. (Thus, if $A = 0$, B can be an arbitrary torsion-free group.)

[1]) The numbers in brackets refer to the Bibliography at the end of this article.

Remark. This theorem says that the endomorphism ring $E(G)$ of a torsion group G is torsion-free if and only if G is the direct sum of groups $C(p_\nu^\infty)$; moreover, in case of a mixed group G, $E(G)$ is torsion-free if and only if G can be written in the form (1), where A is a direct sum of groups $C(p_\nu^\infty)$ and B is a torsion-free group closed for each prime p which is actual for A.

Proof of the necessity of the conditions in the Theorem. Suppose that G is an abelian group with torsion-free endomorphism ring $E(G)$, and G contains elements ($\neq 0$) of finite order. Then all elements of finite order in G form a subgroup A of G, called *the torsion subgroup of* G. Let p be an actual prime for A. Then we state that

$$pG = G. \tag{2}$$

Indeed, if we assume $pG \neq G$, then

$$G \sim G/pG \sim C(p), \tag{3}$$

since the factor group G/pG is an elementary p-group, (i. e., each element $\neq 0$ in G/pG is of order p), and thus a direct sum of groups $C(p)$. Now, since G contains a subgroup $C(p)$, an endomorphism of G can be defined which maps G onto this subgroup $C(p)$. But in this case $E(G)$ cannot be torsion-free since it contains an element of additive order p. Thus (2) is proved.

Now, (2) implies that the p-primary component of the torsion subgroup A of G is closed for p, i. e. this p-primary component is algebraically closed. This is true for each prime p which is actual for G, so that we have shown that A itself is algebraically closed. Then, by a well-known theorem of BAER [1], A is a direct summand of G, i. e., the representation (1) holds with a suitable torsion-free subgroup B of G. Moreover, (2) implies

$$pG = pA + pB = A + pB = G = A + B,$$

i. e., $pB = B$ for each prime p actual for A. So we have proved the necessity of the conditions in the Theorem.

Proof of the sufficiency of the conditions in the Theorem. Suppose that G is an abelian group for which the representation (1) holds with an algebraically closed torsion group A and a torsion-free group B closed for each actual prime for A. We prove that in this case the endomorphism ring $E(G)$ of G is torsion-free.

We show that the endomorphic image of G under an arbitrary endomorphism $\neq 0$ of G, is a subgroup of G the orders of the elements in which are not bounded.[2]) In case $A = 0$ this is trivial. If $A \neq 0$, then let p be an actual prime number for A. Then, by hypothesis, $pA = A$ and $pB = B$, so that

$$pG = p(A + B) = A + B = G.$$

Hence the group G is closed for p. Therefore the endomorphic image of G

[2]) Here ∞ is considered as a „number" greater than each natural number.

is also closed for p under each endomorphism of G. Since this is true for each prime p which is actual for G (thus a fortiori for each p which is actual for an endomorphic image of G), the orders of the elements in the endomorphic image under consideration cannot form a bounded set. This completes the proof.

Bibliography.

[1] R. Baer, The subgroup of the elements of finite order of an abelian group. *Ann. of Math. (Princeton)* (2), **37** (1936), 766—781.

[2] T. Szele, Ein Analogon der Körpertheorie für abelsche Gruppen. *Journal f. d reine u. angew. Math.* **188** (1950), 167—192.

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