A note on the solution of Heun's differential equation in a special case.

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1. The differential equation of HEUN which is of the second order and of FUCHS' type has three singularities in the finite and one in the infinite. It may be written in the form

$$(1)\qquad \begin{gathered} x(x-y)(x-z)\frac{d^2u}{dx^2}+ \\ [A(x-y)(x-z)+Bx(x-z)+Cx(x-y)]\frac{du}{dx}+(Dx+E)u=0 \end{gathered}$$

where the quantities $y, z, A B, \ldots, E$ do not depend on x.

It is customary to write equation (1) by taking $z=1$ which is no loss of generality. However for our present purpose the form (1) is more adapted.

In the following there will be given an example of a function which satisfies a special case of (1) and two other related equations.

Some twelve years ago E. FELDHEIM investigated the function

$$(2)\qquad v=(1-z)^{-b}F\left(a, b, c, -\frac{xz}{1-z}\right)$$

where $F(a, b, c, \xi)$ is the hypergeometric function of GAUSS[1]). He gave an expansion of v in a power series of z:

$$v=\sum_{n=0}^{\infty}\frac{b(b+1)\ldots(b+n-1)}{n!}F(a,-n,c,x)z^n \qquad (|z|<1).$$

It can be easily shown that v as function of z satisfies a differential equation of HEUN's type with singularities at $0, 1, \frac{1}{1-x}$ and ∞. We do not write down this equation. Instead of this we will occupy ourselves rather

1) E. FELDHEIM: A Jacobi-polinomok elméletéhez. *Matematikai és fizikai lapok*, **48** (1941), 453—504, esp. p. 460.

with the function

$$w = y^{-b}(z-x)^{-b} F\left(a, b, c, \frac{z-y}{y} \frac{x}{z-x}\right)$$

which is closely related to the function v[2]).

2. The function w has remarkable properties of symmetry. For seeing it, it is convenient to introduce the quantities $\alpha, \alpha', \alpha''$ defined by the equations

$$a = \frac{\alpha - \alpha' + \alpha''}{2},\ b = \frac{\alpha + \alpha' + \alpha''}{2} - 1,\ c = \alpha.$$

Further let us define the quantities $A, B, \ldots, G''$ by the matrix equations:

$$\begin{pmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{pmatrix} = \begin{pmatrix} \alpha & \alpha'' & \alpha' \\ \alpha' & \alpha & \alpha'' \\ \alpha'' & \alpha' & \alpha \end{pmatrix},$$

$$\begin{pmatrix} D & F & G \\ D' & F' & G' \\ D'' & F'' & G'' \end{pmatrix} = \frac{\alpha+\alpha'+\alpha''-2}{4} \begin{pmatrix} \alpha+\alpha'+\alpha'', & -\alpha-\alpha'+\alpha'', & -\alpha+\alpha'-\alpha'' \\ \alpha+\alpha'+\alpha'', & \alpha-\alpha'-\alpha'', & -\alpha-\alpha'+\alpha'' \\ \alpha+\alpha'+\alpha', & -\alpha+\alpha'-\alpha'', & \alpha-\alpha'-\alpha'' \end{pmatrix}.$$

A direct substitution shows that w satisfies the following set of differential equations:

$$\frac{\partial^2 w}{\partial x^2} + \left[\frac{A}{x} + \frac{B}{x-y} + \frac{C}{x-z}\right]\frac{\partial w}{\partial x} + \frac{Dx+Fy+Gz}{x(x-y)(x-)z} w = 0, \tag{3}$$

$$\frac{\partial^2 w}{\partial y^2} + \left[\frac{A'}{y} + \frac{B'}{y-2} + \frac{C'}{y-x}\right]\frac{\partial w}{\partial y} + \frac{D'y+F'z+G'x}{y(y-z)(y-x)} w = 0, \tag{3'}$$

$$\frac{\partial^2 w}{\partial z^2} + \left[\frac{A''}{z} + \frac{B''}{z-x} + \frac{C''}{z-y}\right]\frac{\partial w}{\partial z} + \frac{D''z+F''x+G''y}{z(z-x)(z-y)} w = 0, \tag{3''}$$

which are all of HEUN's *type.*[3])

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[2]) If $-b > 0, a > 0, c-a > 0$, it is readily shown using EULER's expression of the hypergeometric function that

$$w = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}[(z-x)y - t(z-y)x]^{-b} dt.$$

It is clear from this representation that if b is a non-positive integer then w is a polynomial in x, y and z, further that the substitution $x \to z-x, y \to z-y$ leads to another function of the same type.

[3]) It may be noted that the function satisfies also the following partial differential equations of firts order

$$x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial y} + \left(\frac{\alpha+\alpha'+\alpha''}{2} - 1\right)(x+y+z)\,w = 0$$

and

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} + (\alpha+\alpha'+\alpha''-2)\,w = 0$$

which are symmetric in the three variables.

The quantities y and z are the singular points of equ. (3) i. e. mere parameters of this equations. It is remarkable that if w is regarded as a function of either of the singularities y or z, the function w satisfies an ordinary differential equation of the same HEUN's type.

3. There are other cases too of a set of partial differential equations which have the same property. The solutions of these systems satisfy several ordinary differential equations the singularities of the one being variables of the other and moreover the ordinary differential equations having the same structure.

The simplest example is the partial differential equation of two variables investigated by RIEMANN, DARBOUX and APPELL [4])

$$(x-y)\,u_{xy}-\beta' u_x+\beta u_y=0 \tag{4a}$$

which has a homogeneous solution of any arbitrary degree $-\alpha$ satisfying *Euler*'s differential equation of homogeneity

$$xu_x+yu_y+\alpha u=0. \tag{4b}$$

Eliminating the derivatives u_y and u_{xy} from the set (4a) and (4b) we get

$$x(x-y)u_{xx}+[(\alpha+\beta+1)x-(\alpha-\beta'+1)y]\,u_x+\alpha\beta u=0$$

which is a differential equation of the hypergeometric type in the variable x the singularities being at $0, y$ and ∞.

If we eliminate from (4a) and (4b) the derivatives u_x and u_{xy} the result is another differential equation of the same hypergeometric type owing to the symmetry of the system (4). The variable will be now y and the singularities are at $0, x$ and ∞.

We see that the function u satisfies two ordinary differential equations the singularity of the one being the variable of the orther.

4. Another example is the set

$$\left\{\begin{aligned} (x-y)\,u_{xy}-\beta'\,u_x+\beta\,\,u_y&=0\\ (y-z)\,u_{yz}-\beta''u_y+\beta'\,u_z&=0\\ (z-x)\,u_{zx}-\beta\,\,u_z+\beta''u_x&=0\\ xu_x+yu_y+zu_z+\alpha u&=0\end{aligned}\right. \tag{5}$$

u being a function of x, y and z. The system is compatible, for a direct substitution shows that a solution of it is

$$u=F_1\left(\alpha,\beta,\beta',-\beta''+\alpha+1,\frac{x}{z},\frac{y}{z}\right)z^{-\alpha}$$

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[4]) RIEMANN: Über die Fortpflanzung ebener Luftwellen. *Gesammelte Werke*, pp. 156—175.
DARBOUX: Théorie des surfaces. Vol. 2, p. 81.
APPELL: Sur une équation aux dérivées partielles. *Bull. Sci. Math.*, **6** (1882), 314—318.

where

$$F_1(a, b, b', c, \xi, \eta) =$$
$$= \sum_{m, n=0}^{\infty} \frac{a(a+1)\ldots(a+m+n-1)\cdot b(b+1)\ldots(b+m-1)\cdot b'(b'+1)\ldots(b'+n-1)}{c\cdot(c+1)\ldots(c+m+n-1)\, m!\, n!} \xi^m \eta^n$$

is one of APPELL's hypergeometric functions of two variables.

Incidentally the set (5) of equations seems to be the most simple and symmetric of the various sets of partial differential equations satisfied by APPELL's function F_1. This is a consequence of introducing the homogeneous variables x, y, z instead of ξ and η.

APPELL's function F_1 is known to satisfy an ordinary third order differential equation.[5])

Eliminating the y and z derivatives from the system (5) we get

(6) $$x(x-y)(x-z)\, u_{xxx} + K(x, y, z)\, u_{xx} + L(x, y, z) u_x + Mu = 0$$

where $K(x, y, z)$ and $L(x, y, z)$ are polynomials in x, y and z. Equation (6) is satisfied by each solution of (5).

From the complete symmetry of the system (5) with respect to x, y and z it follows that solutions of the system satisfy also ordinary differential equations of the form

(6′) $$y(y-z)(y-x)\, u_{yyy} + K'(y, z, x)\, u_{yy} + L'(y, z, x)\, u_y + M'u = 0$$

and

(6″) $$z(z-x)\, z-y)\, u_{zzz} + K''(z, x, y)\, u_{zz} + L''(z, x, y)\, u_z + M''\, u = 0$$

where $K', \ldots, L''$ are again polynomials.

Here again it is seen by comparing the last three equations that the independent variable of one of them is a singular point of the others.

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5) APPELL—KAMPÉ DE FÉRIET: Fonctions Hypergéométriques. Paris, 1926, p. 75.