On close-packings of spheres in spaces of constant curvature.

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In the closest packing of non-overlapping equal circles of the common plane each circle is touched by six others[1]). The density of this packing equals $\pi/\sqrt{12} = 0{,}9069\ldots$. Spheres on these circles as equators form a close layer of spheres. Of such layers a packing of equal spheres can be made up in which each sphere is surrounded by twelve others. Such a packing has a density equal to $\pi/\sqrt{18} = 0{,}74048\ldots$. We have good reasons to suppose that no regular or irregular arrangement of equal spheres can have a greater density. But no attempt has so far succeeded in proving this conjecture.

A new point of view arises by considering the problem in spaces of constant curvature. The aim of the present paper is to point out what beautiful results can be expected in this direction.

Let us start by recapitulating the analogous results[2]) on surfaces of constant curvature, i. e. on the surface of a common sphere, or on the Euclidean plane or on the hyperbolic one. Let us consider on a surface of a constant curvature k a system of at least three non-overlapping circles of radius r. Then the surface can be divided into triangles so that the density of the system in each triangle is $\leqq d$ where d denotes the density of three circles of radius r mutually touching one another in the triangle t determined by their centres. Obviously d depends only on the value kr^2, so that we may write $d = d(kr^2)$. The density D of the circles, on the whole surface, can be defined by the arithmetic mean of the densities in the triangles in question weighted by the areas of the triangles. So we have

$$D \leqq d(kr^2). \tag{1}$$

Let us denote an angle of t by $2\pi/N$. The number $N = N(kr^2)$ (which is not necessarily an integer) can be interpreted as the „number" of circles of radius r we can put on a circle of radius r on a surface of curvature k. On the sphere we have $2 \leqq N < 6$, in the Euclidean plane $N = 6$ and in the hyperbolic one $N > 6$. If $N = 2, 3, \ldots$ then there exists a regular decomposition

[1]) An account of this range of problems can be found in the book of the author: Lagerungen in der Ebene, auf der Kugel und im Raum. Berlin—Göttingen—Heidelberg, 1953.

[2]) L. FEJES TÓTH, Kreisausfüllungen der hyperbolischen Ebene. *Acta Math. Acad. Sci. Hung.* 4 (1953), 103—110.

of the surface of Schläfli symbol $\{3, N\}$ whose faces are congruent to t. The densest packing of circles arises in this case by placing the centres in the vertices of this $\{3, N\}$. The circles are then the incircles of the faces of the dual decomposition $\{N, 3\}$. The density of this arrangement is $D = d(kr^2)$.

If $N \to \infty$, $d(kr^2)$ tends to $3/\pi$ increasingly, so that we have independently of kr^2

$$D \leqq \frac{3}{\pi} = 0{,}955\ldots .$$

The density $3/\pi$ can be reached by an arrangement of horocircles centred at the vertices of the tessellation $\{3, \infty\}$ which plays a rôle in the theory of modul functions.

Let us now turn to the problem in the space. By a (three-dimensional) space of constant curvature we mean the surface of a four-dimensional Euclidean sphere, or the Euclidean space, or the hyperbolic one. We have the following conjecture. Consider in a space of constant curvature k at least four non-overlapping spheres of radius r. Then the space can be decomposed into tetrahedra so that the density of the spheres in each tetrahedron is $\leqq \delta(kr^2)$, where $\delta(kr^2)$ denotes the density of four spheres mutually touching one another in the tetrahedron τ determined by the centres of the spheres. If we define the density Δ of the spheres in the whole space by the arithmetic mean of the densities in the tetrahedra weighted by their volumes, then we have by our conjecture

(2) $$\Delta \leqq \delta(kr^2).$$

If the dihedral angles of τ are equal to $2\pi/3$, $2\pi/4$, $2\pi/5$ or $2\pi/6$, the space can be decomposed into tetrahedra congruent to τ so as to form a regular honeycomb $\{3, 3, 3\}$, $\{3, 3, 4\}$, $\{3, 3, 5\}$ or $\{3, 3, 6\}$, respectively. The first three decompositions are central projections of the regular 5-cell (simplex), 16-cell (cross polytope) and 600-cell upon their insphere. The honeycomb $\{3, 3, 6\}$ decomposes the hyperbolic space into tetrahedra of greatest volume. Spheres around the vertices of these honeycombs form, according to our conjecture, a densest packing. In the last case the spheres are horospheres. The density $\bar{\Delta}$ of this packing can be expressed in terms of an interesting series [3]):

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[3]) It can be shown that $\lim\limits_{x \to -\infty} \delta(x) = \frac{\sqrt{3}}{2T}$ where T denotes the greatest volume a tetrahedron can possibly have in the hyperbolic space of curvature -1. We use here the value of T in the form

$$T = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \ldots\right)$$

given by H. S. M. Coxeter, The functions of Schläfli and Lobatschefsky, *Quarterly Journal of Mathematics* **6** (1935), 13—29.

$$\bar{\Delta} = \lim_{x \to -\infty} \delta(x) = \left(1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \cdots\right)^{-1} = 0{,}853\ldots .$$

This is probably the absolute maximum of the density of at least three equal spheres (of finite or infinite radius) in spaces of constant curvature.

The direct way to the above conjecture is analogous to the proof of (1) and requires the solution of the following maximum problem. Let σ be the sum of the solid angles of a tetrahedron τ in a space of constant curvature, all sides of which have a length $\geqq 2r$ and the radius of the circumsphere of which is $\leqq 2r$. Among these tetrahedra we have to find that one which maximises the quotient[4] σ/τ. The best tetrahedron is probably the regular one. Owing to the fact that the Euclidean space cannot be filled out by regular tetrahedra, the densest packing of spheres involves here more complicated figures. This throws light on the fact that the problem of filling out the greatest part of the space by equal spheres (of arbitrarily chosen size) seems in hyperbolic space to be easier to attack than in the Euclidean one. In the Euclidean space our conjecture yields the estimation

$$\Delta < \sqrt{18}\left(\arccos\frac{1}{3} - \frac{\pi}{3}\right) = 0{,}77964\ldots .$$

Instead of the direct way mentioned above we approach our conjecture in an other way and try to support it by two remarks having also some interest in themselves. Both remarks rest on a general theorem[5] which may be stated as follows:

Let $P_1, \ldots, P_n$ be $n \geqq 3$ points of the surface S of a common sphere and let $F(\varrho)$ be a non-decreasing function defined for[6] $0 \leqq \varrho \leqq \frac{1}{2}\sqrt{\pi S}$. Further let $\Delta_P = \min(PP_1, \ldots, PP_n)$ denote the spherical distance of a variable point P of S from the point P_i nearest to it and dS the area element of S at P. Then

$$\frac{1}{S}\int_S F(\Delta_P)\,dS \geqq \frac{1}{s}\int_s F(\lambda_P)\,dS \tag{3}$$

where $s = p_1 p_2 p_3$ denotes an equilateral spherical triangle of area $s = S/(2n-4)$ and $\lambda_P = \min(Pp_1, Pp_2, Pp_3)$ the distance of P from the vertex of s nearest to it.

For a non-increasing function $F(\varrho)$ the inequality holds evidently in opposite sense. Equality holds if (and, in case $F(\varrho)$ is strictly increasing or decreasing, only if) the points $P_1, \ldots, P_n$ are vertices of a $\{3,2\}$, $\{3,3\}$, $\{3,4\}$ or $\{3,5\}$.

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[4]) In what follows we shall denote a set and its content by the same symbol.

[5]) See the book quoted in [1]).

[6]) $\sqrt{\pi S}$ is the length of a greatest circle of S.

Let $S = 4\pi$ and let r be a number such that $0 < r < \pi$. In case the function $F(\varrho)$ is defined by

$$F(\varrho) = \begin{cases} 1 & \text{for} \quad 0 \leqq \varrho < r \\ 0 & \text{for} \quad r \leqq \varrho \leqq \pi, \end{cases}$$

our inequality yields

$$\frac{T}{S} \leqq \frac{t}{s}$$

where T and t denote that part of S and s, respectively, which is covered by the circles $C_1, \ldots, C_n$ and c_1, c_2, c_3, respectively, of radius r centred at $P_1, \ldots, P_n$ and p_1, p_2, p_3, respectively.

Since the sum of the angles of s equals $s + \pi$, the density of c_1, c_2, c_3 in s is

$$\frac{s+\pi}{2\pi} C : s = \frac{C}{2\pi}\left(1 + \frac{\pi}{s}\right) = \frac{Cn}{4\pi}; \; C = C_1 = c_1.$$

This is just the density of $C_1, \ldots, C_n$ on S.

Let us now suppose that the circles $C_1, \ldots, C_n$ do not overlap. Since in this case the density computed just now is T/S, it follows from the last inequality that the sum of the areas of the circular sectors sc_1, sc_2, sc_3 is less than, or equal to, the area of their union t. Consequently, the circles c_1, c_2, c_3 do not overlap, so that $2r$ cannot be greater than the lenght of a side of s.

The result obtained can be formulated as follows: From among $n \geqq 3$ points of a sphere of surface area 4π there can alvays be selected two, having a spherical distance

$$\leqq \arc\cos \frac{\cot^2 \omega_n - 1}{2} \tag{4}$$

where

$$\omega_n = \frac{n}{n-2} \frac{\pi}{6}$$

is a half angle of s. This theorem, which can also be proved in various ways directly, is equivalent to (1) in case $k > 0$.

Now we are able to estimate the maximal number n of spheres of radius r in a space of constant curvature k we can put on a sphere of radius r. Let 2α be an angle of the triangle determined by the centres of three spheres of radius r touching one another. We have

$$\sin \alpha = \frac{\sin \sqrt{k} r}{\sin 2\sqrt{k} r} = \frac{1}{2 \cos \sqrt{k} r}.$$

Evidently, n equals the number of points we can place on a sphere of surface area 4π so that any two points have a spherical distance $\geqq 2\alpha$. Hence by (4)

$$2 \cos 2\alpha \geqq \cot^2 \omega_n - 1.$$

Expressing this inequality in terms of r we have our first remark:

The maximal number n of spheres of radius r of a space of constant curvature[7] $k \leqq r^{-2}(\operatorname{arc\,tan}\sqrt{2})^2$ *which can be put on a sphere of the same size satisfies the inequality*[8])

$$\cot^2\omega_n + \tan^2\sqrt{k}\,r \leqq 2.$$

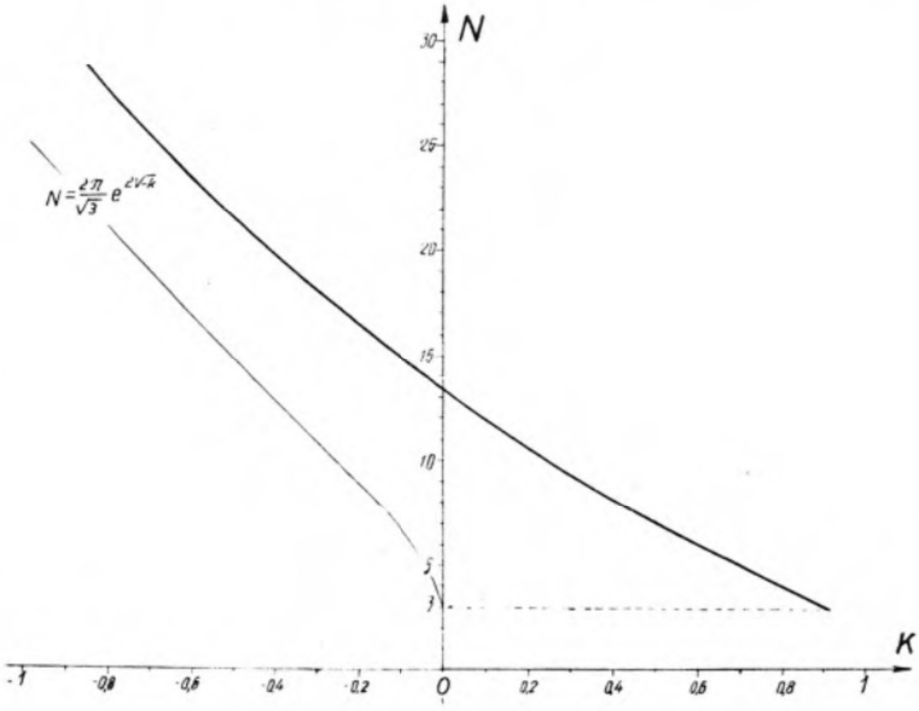

Fig 1.

Defining $N = N(kr^2)$ by

$$\cot^2\omega_N + \tan^2\sqrt{k}\,r = 2,$$

our inequality takes the form

$$n \leqq N. \tag{5}$$

Considering spheres of unit radius, $N = N(k)$ can be interpreted as the „number" of unit spheres in a space of curvature k arranged „ideally close" on a unit sphere[9]. Fig 1. shows the function $N = N(k)$. If $N = 3, 4, 6$ or 12, our inequality is exact. The spheres touch in these cases the inner one in the vertices of a regular triangular net $\{3,2\}$, $\{3,3\}$, $\{3,4\}$ or $\{3,5\}$, i. e. in the vertices of a regular triangle inscribed in a greatest circle, of a regular tetrahedron, octahedron or icosahedron, respectively. Furthermore,

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[7]) The restriction $kr^2 \leqq (\operatorname{arc\,tan}\sqrt{2})^2$ comes from the supposition $n \geqq 3$ in (4).

[8]) This inequality is equivalent to $\cot^2\omega_n - \tanh^2\sqrt{-k}\,r \leqq 2$.

[9]) In the case of Euclidean space (5) yields $n < N(0) = 13{,}4\ldots$. See: K. Schütte and B. L. van der Waerden, Das Problem der dreizehn Kugeln. *Math. Ann.* **125** (1953), 325—334. There is shown that $n = 12$.

(5) gives the exact asymptotic estimation for $N \to \infty$, namely

$$\overline{\lim_{N\to\infty}} \frac{n}{e^{2\sqrt{-k}r}} \leq \lim_{N\to\infty} \frac{N}{e^{2\sqrt{-k}r}} = \frac{2\pi}{\sqrt{3}}.$$

The tangent points of the extremal arrangement can be considered in the limit case as the vertices of a regular net $\{3, 6\}$ on a horosphere.

Our second remark concerns the n-hedron of minimal volume containing a given sphere S in a space of constant curvature. Let a be a plane touching S at the point A and U a convex domain in a. Further let u be the central projection of U from the centre O of S and H the convex hull of O and U. Obviously, the volume of H can be expressed in the form

$$H = \int_u F(AP)\,dS$$

where $F(\varrho)$ is an increasing function[10]). Applying the inequality (3) to this function we get the following theorem:

Let V be a convex n-hedron in a space of constant curvature containing a sphere of surface S. Further let s be a regular spherical triangle on S of area $s = S/(2n-4)$ and v the hexahedron bounded by the three planes through the sides of s and the three planes touching S at the vertices of s. Then

(6) $$V \geq (2n-4)v$$

and equality holds only if V is a regular tetrahedron, hexahedron or dodecahedron circumscribed to S.

The minimum property of the regular polyhedra with trihedral vertices according to which they have the least volume among the polyhedra of the same insphere and the same number of faces, was already known in Euclidean space. Now we see that this minimum property is preserved in spaces of constant curvature.

Let us now return to our packing problem. If $k > 0$, the problem is equivalent to the following one: to find on the surface of a unit sphere of the four-dimensional Euclidean space such an arrangement of m points in which the least spherical distance of two points reaches its maximum $2r_m$. Spheres of radius r_m around the points of a such arrangement form a densest packing.

The best arrangements are for $m = 2, 3, 4$ and 5 two antipodal points, the vertices of a regular triangle inscribed in a greatest circle, the vertices of a regular tetrahedron inscribed in a greatest sphere and the vertices of a regular 5-cell, respectively. The best arrangements of 6 and 7 points are not unique and we have $r_6 = r_7 = r_8$. For $m = 8$ the best arrangement coin-

[10]) $S.F(AP)$ is equal to the volume of the sphere of radius Op where p is the projection of P upon a.

cides with the vertices of a regular 16-cell. Our statements concerning the cases $m=6,7$ and 8 follow from a more general result of HAJÓS and DAVENPORT[11]).

For no other number of points the best configurations are known. However, it may be supposed (in accordance with our conjecture) that for $m=120$ the points must be distributed in the vertices of a regular 600-cell. The following table shows the values of $r=r_m$, N and the approximative values of the density

$$\Delta=\frac{m}{2\pi}(2r_m-\sin 2r_m)$$

in the cases mentioned above. For $m=2$ and 3 the values of N have been determined by a natural convention.

m	r	N	Δ
2	$90°$	1	1
3	$60°$	2	0,587
4	$\operatorname{arc\,tan}\sqrt{2}\approx 54°44'\,8''$	3	0,616
5	$\operatorname{arc\,sin}\sqrt{\frac{5}{8}}\approx 52°14'20''$	4	0,681
6	$45°$	6	0,545
7	$45°$	6	0,636
8	$45°$	6	0,727
120	$\operatorname{arc\,sin}\frac{\sqrt{5}-1}{4}=18°$	12	0,774

In what follows we allow the curvature k of the space to be positive or negative, but we restrict our attention to values of kr^2 for which N is an integer $\geqq 3$. Let $O_1, O_2, \ldots$ be the centres of the spheres $S_1, S_2, \ldots$ of such a packing. Consider the set Γ_i of the points of the space whose distance from O_i is less than, or equal to, the distance from any other centre O_j $(j\neq i)$. Γ_i is a convex polyhedron (finite or infinite) containing S_i. We shall call it the cell of S_i. The cells $\Gamma_1, \Gamma_2, \ldots$ cover the space simply and without gap.

Suppose first that the number of the faces of all cells is $\leqq N$. Then by (6) we have $\Gamma_i \geqq (2N-4)v$, i. e.

$$\Delta_i=\frac{S_i}{\Gamma_i}\leqq\frac{S_i}{2N-4}:v.$$

Since the hexahedron v has a solid angle at O_i equal to $4\pi/(2N-4)$, the right side of the last inequality equals the density of S_i in v. On the other

11) See Problem 35, Matematikai Lapok **2** (1951), p. 68.

hand, the face angles of v at O_i are equal to an angle of an equilateral triangle of side $2r$. Four such hexahedra can be made up to form a regular tetrahedron of side-length $2r$. Consequently, the density considered just now is nothing else than the density $\delta(kr^2)$ defined in connection with the inequality (2). Thus

$$\Delta_i \leqq \delta(kr^2).$$

Defining Δ as a mean-value of the densities Δ_i of the packing in the single cells we have a fortiori $\Delta \leqq \delta(kr^2)$.

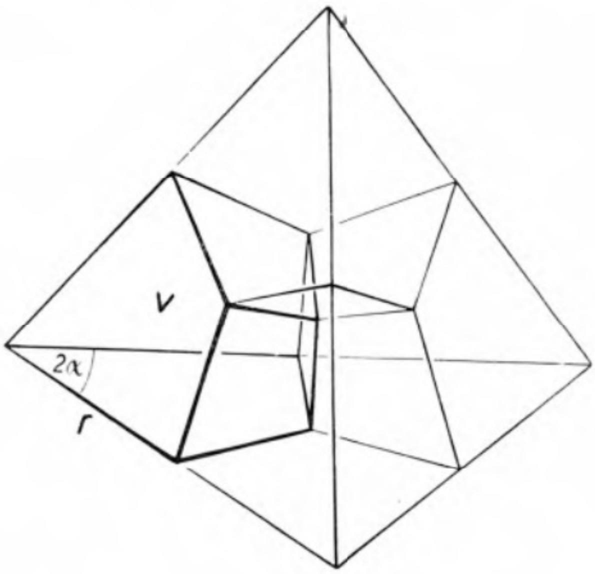

Fig. 2.

Our discussions show especially: *Among all arrangements of* 120 *points on the surface of a four-dimensional sphere with at most dodecahedral cells, the system of the vertices of the regular* 600*-cell is the best one.* Let us move the vertices of the 600-cell in a little neighbourhood of their original position. Since the cells of the moved vertices do not cease to be dodecahedra, the system of vertices of the 600-cell has been recognised as a locally best distribution: *the polytope Π' arising from the regular* 600*-cell Π by little motions of their vertices on his circumsphere has a shorter edge than Π, unless Π' and Π are congruent.*

We proceed now by considering the case of a cell Γ_i with more than N faces. Since by (5) S_i can be surrounded by at most N spheres, Γ_i cannot be circumscribed to S_i and thus its volume cannot be small. This support the conjecture that the inequality $\Delta_i \leqq \delta(kr^2)$ holds in any case.

In the following somewhat more detailed discussion of the case $N = 12$ we shall make use of a sharpening of (6) which, in the case of a dodecahedron, reads as follows: In a space of constant curvature let $\bar{D}$ be a regular dodecahedron of insphere s and circumsphere S. Then among all dodecahedra D containing s, the volume of the common part of D and S assumes its minimum for the regular one:

$$DS \geqq \bar{D}. \tag{7}$$

The proof is the same as that of (6) using instead of $F(\varrho)$ the function which arises from $F(\varrho)$ by replacing the values $F(\varrho)$ greater than S/f by S/f where f denotes the surface area of s (which was denoted in the proof of (6) by S).

If $N=12$ and $k=1$, we have

$$r = \text{arc sin} \frac{\sqrt{5}-1}{4} = 18^\circ.$$

Let us compute the radius R of the circumsphere S of the regular dodecahedron of insphere-radius r:

$$R = \text{arc cos} \frac{1}{4} \sqrt{7+3\sqrt{5}} = 22^\circ 14'.$$

Since a sphere s of radius r can be touched by at most 12 others, among the centres $O, O_1, \ldots, O_{13}$ of s and 13 further spheres we must have a relation of the form

$$\frac{1}{13}\sum_{i=1}^{13} OO_i \geqq c$$

with a constant $c > 2r = 36^\circ$. The demonstration of the inequality

$$\frac{1}{13}\sum_{i=1}^{13} OO_i \geqq \frac{24r+2R}{13} \approx 36^\circ 39' \tag{8}$$

would complete the proof of the extremal property of the 600-cell.

Let us control this inequality, for example, in case $OO_1 = \ldots = OO_{12} = 2r$. Then $OO_{13} \geqq 2a$ where a denotes the altitude of the tetrahedron τ of edge-length $2r$. Since $a = 30^\circ$, the mean-value in question is greater than, or equal to, $\frac{24r+2a}{13} \approx 37^\circ 51'$.

We can also compare (8) in case $OO_1 = \ldots = OO_{13}$ with a conjecture of SCHÜTTE[12]), according to which $OO_1 = \ldots = OO_{13} > 40^\circ 15'$.

Both constans are essentially greater than the constant in (8). Thus (8) seems to be satisfied in abundance.

Suppose now that the cell Γ of s has more than twelve faces of which those generated by $O_1, \ldots, O_{13}$ have points with S in common. Let us move $O_1, \ldots, O_{13}$ on the half-lines $OO_1, \ldots, OO_{13}$ under the conditions $OO_i \geqq 2r$

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[12]) See K. SCHÜTTE and B. L. VAN DER WAERDEN, Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz? *Math. Ann.* **123** (1951), 96—124.

($i = 1, \ldots, 13$) and the condition of (8). Then it is not difficult to show that ΓS assumes its minimum in the case when, say, $OO_1 = \ldots = OO_{12} = 2r$ and $OO_{13} \geqq 2R$. Thus the problem reduces to the case of dodecahedral cells and the solution is given by (7).

As far as I know no extremum property of the three "non trivial" regular polytopes $\{3, 4, 3\}$, $\{3, 3, 5\}$ and $\{5, 3, 3\}$ is yet known. We hope to contribute, with our above remarks, to the beginning of a systematic study of the regular polytopes from this point of view.

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