Some saddle-points in $\Lambda \otimes \bar{\Lambda}$.

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§ 1. Lattice saddle-points.

Let Γ be a complete lattice[2]) with partial ordering $\leqq$. Let p and q be two properties each of which is relevant to the members of Γ (that is: dichotomous in Γ). We call such a property a *lower property (upper property)* in Γ provided the set of all elements of Γ which have the property is non-null and contains all predecessors (successors) of any of its elements. If p is a lower property in Γ and q is an upper property in Γ, we say that an element of Γ, with respect to p and q, is a saddle-point if it has both properties and is maximal with respect to having property p and minimal with respect to having property q. If there is at least one element of Γ having both properties and if every such element is a saddle-point, we say that p and q are *saddle-point properties*. Clearly, this amounts to saying that the set of elements having property p and the set of elements having property q intersect in a non-null, totally unordered subset of Γ.

§ 2. The topolattice.

Let S be an infinite set. The topolattice[3]) Λ of S is the set of all T_1 topologies definable in S with the partial ordering: $\alpha \leqq \beta$ if and only if every subset of S open under β is open under α. As is well known[4]), Λ is a complete lattice.

We denote by $\bar{\Lambda}$ the lattice obtained by dualizing the ordering of Λ. We denote by $\Lambda \otimes \bar{\Lambda}$ the direct product (cardinal product; cf.[2])) of the

[1]) Presented to the American Mathematical Society; Christmas, 1953.

[2]) GARRETT BIRKHOFF, Lattice theory. Revised ed. (American Mathematical Society, New York, 1948.)

[3]) ROBERT BAGLEY and DAVID ELLIS, On the topolattice and permutation group of an infinite set. (In preparation.)

[4]) R. VAIDYANATHASWAMI, Treatise on Set Topology. (Indian Mathematical Society, Madras, 1947.)

lattices Λ and $\bar{\Lambda}$. As is well known[4]) the property of being T_2 (HAUSDORFF) is a lower property in Λ and the property of being bicompact is an upper property in Λ. In fact, we have the

Theorem (VAIDYANATHASWAMI). *The properties of being T_2 and of being bicompact are saddle-point properties in Λ.*

To actually obtain a class of these saddle-points other than familiar examples such as the Euclidean topology of a closed interval of real numbers we may take $s \in S$ and let τ_s be that topology in S in which all sets are open except those which have s as a member and whose complements are infinite.

§ 3. Permutation saddle-points.

Let f be any permutation of the set S. We consider the properties for (σ, τ) in $\Lambda \otimes \bar{\Lambda}$:

Property p: The mapping f is continuous when σ is taken as domain opology and τ as range topology.

Property q: The mapping f is an open mapping when σ is taken as domain topology and τ as range topology.

It is obvious that p and q are, respectively, a lower property and an upper property in $\Lambda \otimes \bar{\Lambda}$. We assert the

Theorem. *The Properties p and q just defined are saddle-point properties in $\Lambda \otimes \bar{\Lambda}$.*

Proof. The proof is given in the following two lemmas. First we observe that there is a class of saddle-points independent of f.[5]) Let τ be any topology in S defined as follows: there is a cardinal number m which is infinite and a proper subset G of S is open under τ if and only if its complement G^* has a cardinal number less than m. Then (τ, τ) has properties p and q

Lemma 1. *If (σ, τ) has properties p and q and if $(\alpha, \beta) > (\sigma, \tau)$, then (α, β) does not have property p.*

Proof. Assume the hypotheses. Now, $\alpha \geqq \sigma$ and $\beta \leqq \tau$ and one of these inequalities is strict. Suppose first that $\alpha > \sigma$. Let G be a set open under σ and not open under α. Then $f(G)$ is open under τ, since (σ, τ) has property q, and is open under β, since $\beta \leqq \tau$. But $G = f^{-1}(f(G))$ is not open under α and f fails to be continuous in (α, β). Suppose on the other hand that $\beta < \tau$. Let H be a set open under β and not open under τ. Since H is not open under τ and (σ, τ) has property $q, f^{-1}(H)$ is not open

[5]) Advanced Problem 4548, *Amer. Math. Monthly*, **60** (1953), p. 482.

under σ, and, hence, is not open under α, since $\alpha \geqq \sigma$. Thus, f fails again to be continuous in (α, β).

From a proof dual to that of Lemma 1 we have

Lemma 2. *If* (σ, τ) *has properties* p *and* q *and if* $(\alpha, \beta) < (\sigma, \tau)$ *then* (α, β) *does not have property* q.

Finally we note that if G is a subset of S with $f(G) = G$ and if we take as γ_G the topology in S with G and all complements of finite sets taken as sub-base of open sets, then (γ_G, γ_G) is a saddle-point.

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